From Feynman diagrams to cross sections @ NNLO

William J. Torres Bobadilla
University of Liverpool

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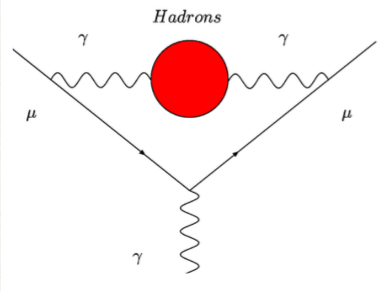
The Spine, Liverpool, UK

Motivation

- MUonE experiment :: new proposal for a_μ^{HVP} with running of $\Delta\alpha_{\text{had}}$

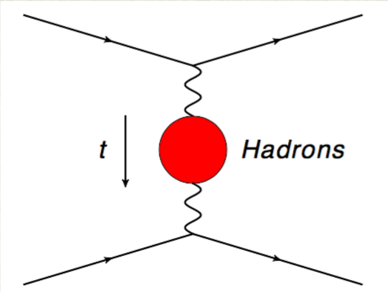
time-like formula



$$a_\mu^{\text{HVP}} = \left(\frac{\alpha m_\mu^2}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds \frac{R_{\text{had}}(s)\, K(s)}{s^2}$$

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)\left(s/m_\mu^2\right)}$$

space-like formula



$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\,\Delta\alpha_{\text{had}}\left[t(x)\right]$$

$$t(x) = \frac{x^2 m_\mu^2}{x-1} < 0$$

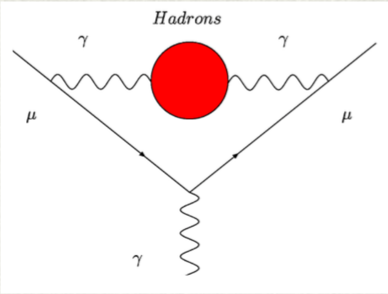
$\Delta\alpha_{\text{had}}[t(x)]$ *can be extracted from the differential cross-section of* $\mu^+e^- \to \mu^+e^-$

$$R_{\text{had}} = \frac{d\sigma_{\text{data}}\left(\Delta\alpha_{\text{had}}\right)}{d\sigma_{\text{MC}}\left(\Delta\alpha_{\text{had}} = 0\right)} \sim 1 + 2\Delta\alpha_{\text{had}}(t)$$

Motivation

- MUonE experiment :: new proposal for a_μ^{HVP} with running of $\Delta\alpha_{\mathrm{had}}$

time-like formula



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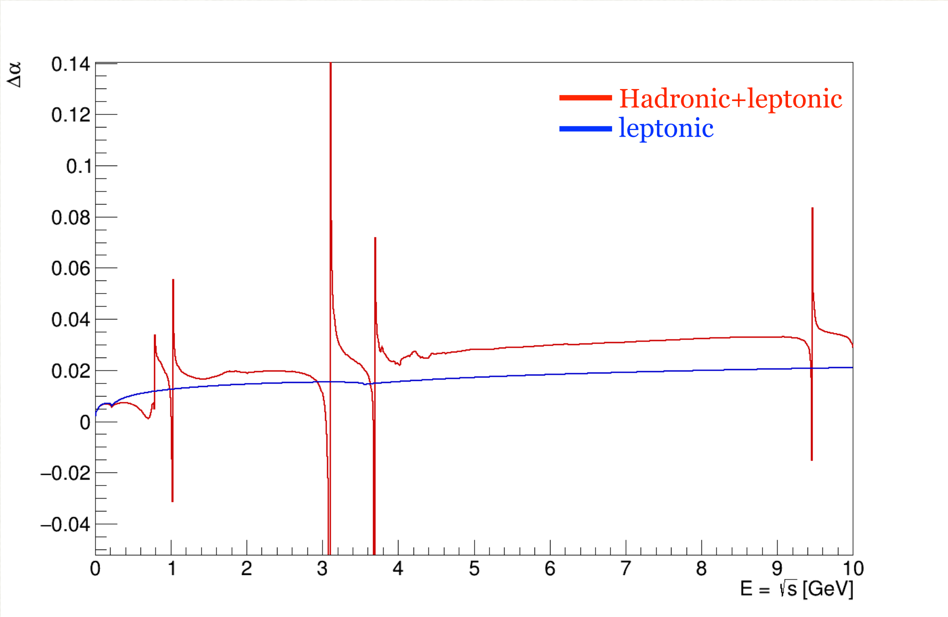
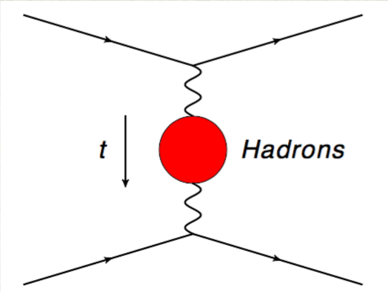
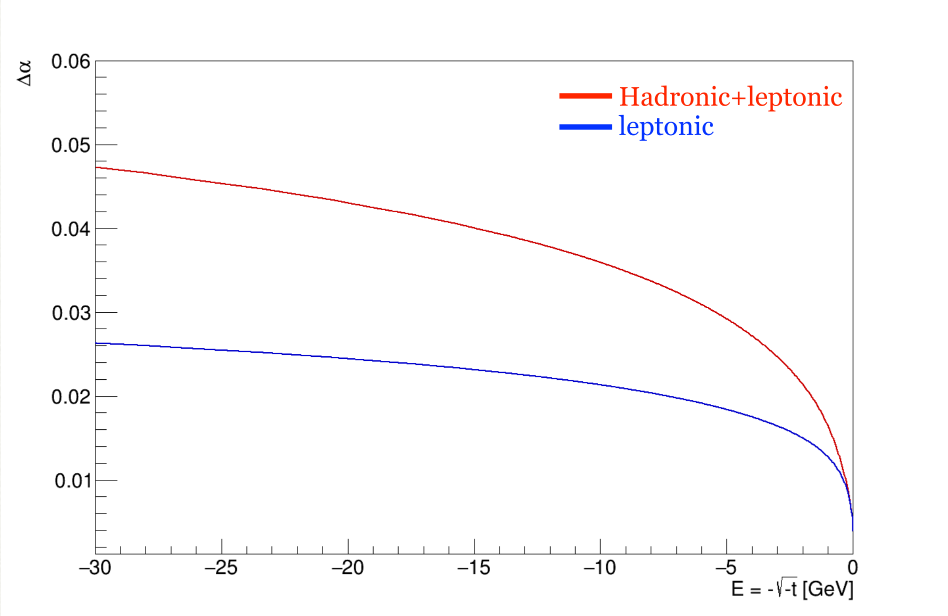


[Pilato's courtesy]

space-like formula



$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\,\Delta\alpha_{\mathrm{had}}[t(x)]$$



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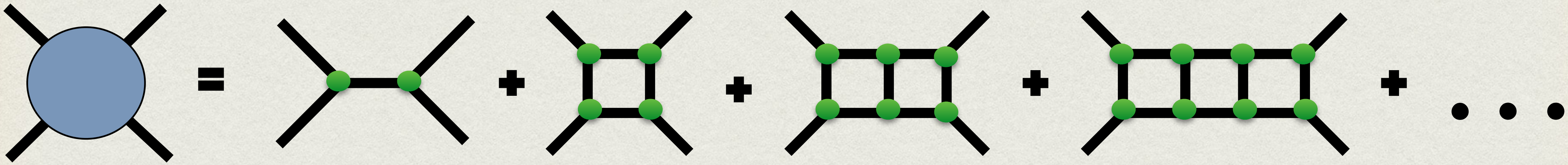
Standard approach @multi-loop level



Draw all Feynman diagrams

Profit from Unitarity based methods

Generate integrands

Profit of DimReg

Use Integration-By-Parts identities

Sector Decomposition
Auxiliary mass flow
Series expansion
Tropical geometry

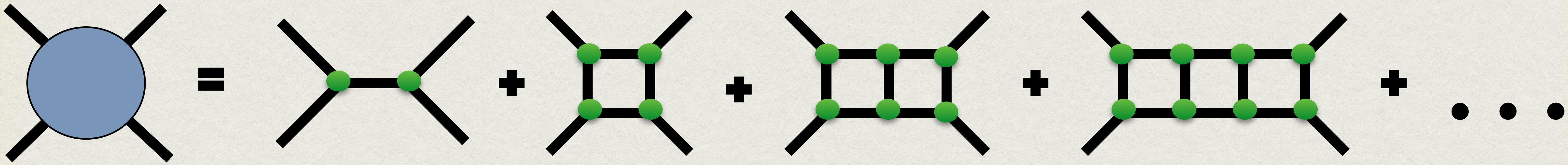
Numerically

Evaluate integrals

Analytically

Diff. Eqs.

Standard approach @multi-loop level



Draw all Feynman diagrams

Profit from Unitarity based methods

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Use Integration-By-Parts identities

Sector Decomposition
Auxiliary mass flow
Series expansion
Tropical geometry

Numerically

LTD approach

Evaluate integrals

Analytically

Diff. Eqs.

Outline

- Motivation
- Analytic evaluation of scattering amplitudes
- Towards (N)NLO w/ Phokhara
- Conclusions & Outlook

$e - \mu$ elastic scattering at NNLO

eμ-scattering @ NLO

$$\mu^+(p_1) + e^-(p_2) \to \mu^+(p_3) + e^-(p_4)$$

Anatomy

- Born matrix element
  tree-level & n-pt process

- Real contribution
  Tree-level (n+1)-particles

- Virtual Contribution
  one-loop n-particles

- No assumption made in the newest result
- QED & EW effects
- Full lepton mass dependence
- Fully differential fixed order MC @ NLO

$$\hat{\sigma}_{NLO} \sim \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^R_{NLO} + \int_{\mathrm{d}\Phi_m} \mathrm{d}\hat{\sigma}^V_{NLO}$$ + MC integration

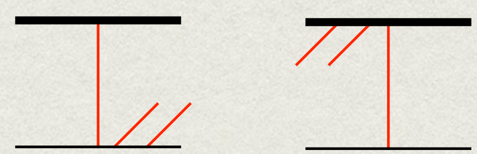
[Carloni Calame, Alacevich, Chiesa, Montagna, Nicrosini, Piccinini (2018)]

eμ-scattering @ NNLO

$$\mu^+(p_1) + e^-(p_2) \to \mu^+(p_3) + e^-(p_4)$$

Anatomy @ NNLO

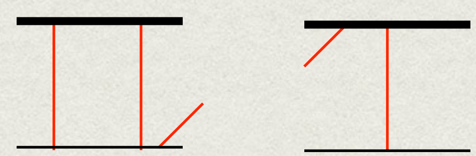
- Real-Real contribution
  Tree-level (*n*+2)-particles



[OpenLoops framework]

<<Adrian's talk

- Real-Virtual Contribution
  one-loop (*n*+1)-particles



- Virtual-Virtual Contribution
  two-loop *n*-particles

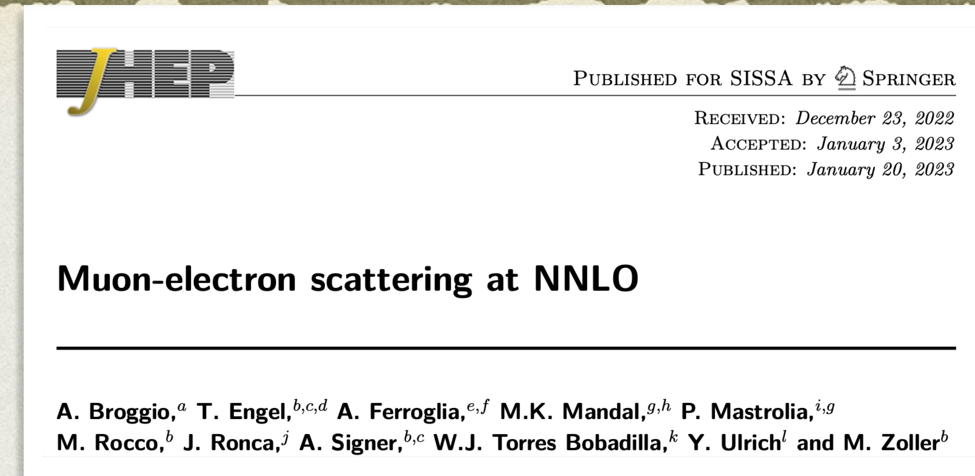
[Bonciani, WJT et al (2021)]

$$\hat{\sigma}_{NNLO} \sim \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

+ Subtractions & MC integrations

[McMule framework]

$e\mu$-scattering @ NNLO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Muon-electron scattering at NNLO

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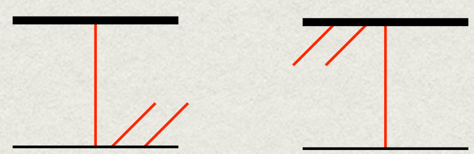
$$\mu^+(p_1) + e^-(p_2) \to \mu^+(p_3) + e^-(p_4)$$

Anatomy @ NNLO

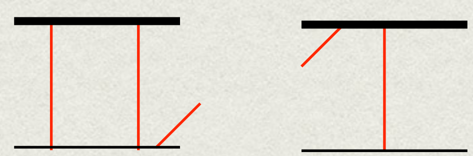
- Real-Real contribution
  Tree-level (n+2)-particles



[OpenLoops framework]

<<Adrian's talk

- Real-Virtual Contribution
  one-loop (n+1)-particles



- Virtual-Virtual Contribution
  two-loop n-particles

[Bonciani, WJT et al (2021)]

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+ Subtractions & MC integrations

[McMule framework]

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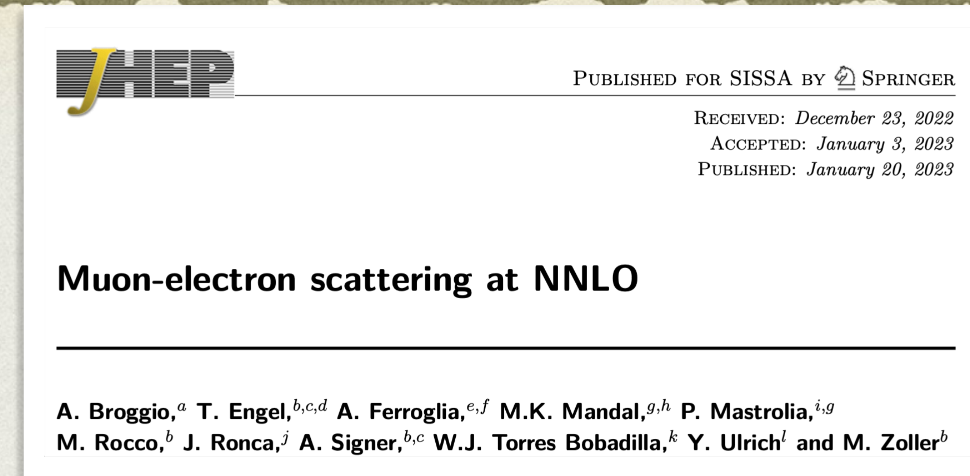
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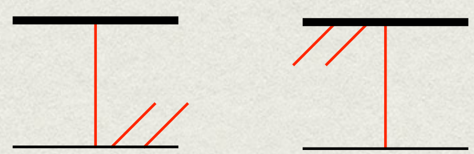
$$\mu^+(p_1) + e^-(p_2) \to \mu^+(p_3) + e^-(p_4)$$

Anatomy @ NNLO

☆ **This talk**

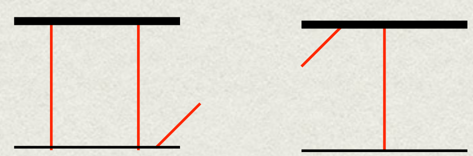
- Real-Real contribution
  Tree-level (n+2)-particles



[OpenLoops framework]

<<Adrian's talk

- Real-Virtual Contribution
  one-loop (n+1)-particles



- Virtual-Virtual Contribution
  two-loop n-particles

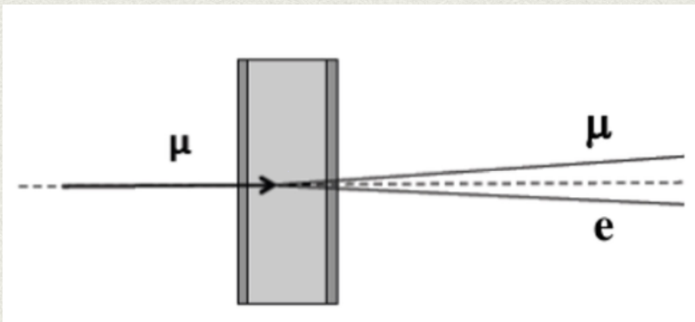
[Bonciani, WJT et al (2021)]

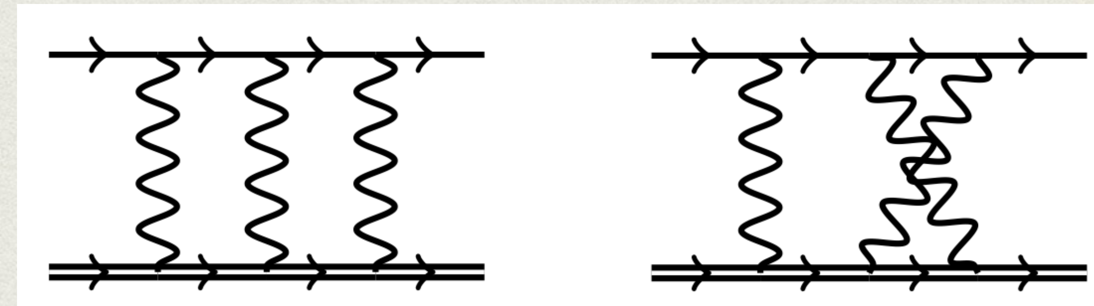
$$\hat{\sigma}_{NNLO} \sim \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

+ Subtractions & MC integrations

[McMule framework]

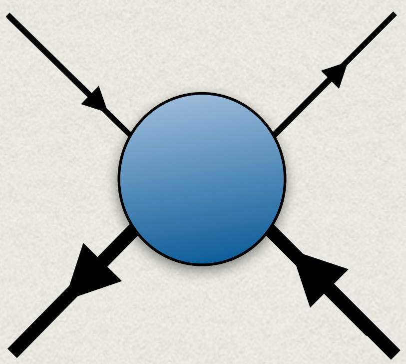
eμ-scattering @ NNLO





- In the massless electron limit ($m_e^2 = 0$)
  4-point process depending on 3 scales

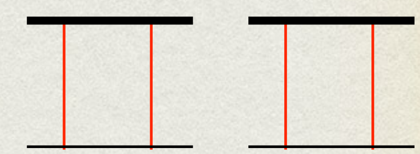
$$s = (p_1 + p_2)^2 \,, \quad t = (p_2 - p_3)^2 \,, \quad u = (p_1 - p_3)^2 \,, \quad s + t + u = 2M^2 \,.$$

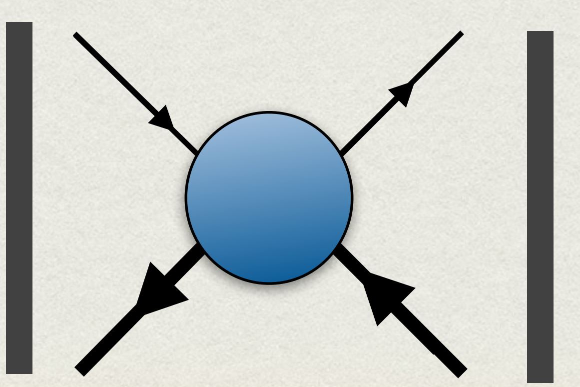


$$\mathcal{A}(\alpha) = 4\pi\alpha \left[\mathcal{A}^{(0)} + \left(\frac{\alpha}{\pi}\right) \mathcal{A}^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 \mathcal{A}^{(2)} + \mathcal{O}\left(\alpha^3\right) \right]$$

- Compute interference

$+$



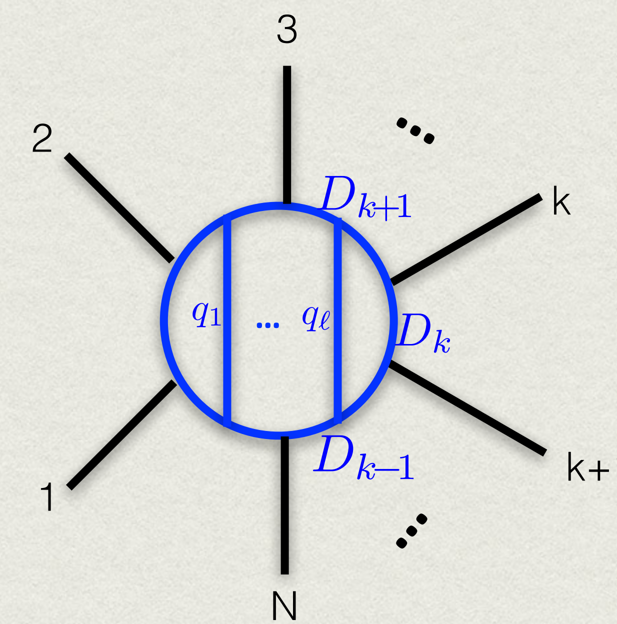


$$\mathscr{M}^{(2)} = \frac{1}{4} \sum_{\text{spins}} 2\mathrm{Re}\left(A_n^{(2)} A_n^{(0)*}\right) + 2\mathrm{Re}\left(A_n^{(1)} A_n^{(1)*}\right)$$

$e\mu$-scattering @ NNLO

- Deal w/ Feynman integrals

$$J_N^{(L),D}(1,\ldots,n;n+1,\ldots,m) = \int \prod_{i=1}^{L} \frac{d^D \ell_i}{\imath\, \pi^{D/2}} \frac{\prod_{k=n+1}^{m} D_k^{-\nu_k}}{\prod_{j=1}^{n} D_j^{\nu_j}}$$



- Integrand/integral reductions

[Chetyrkin, Tchakov]
[Laporta]

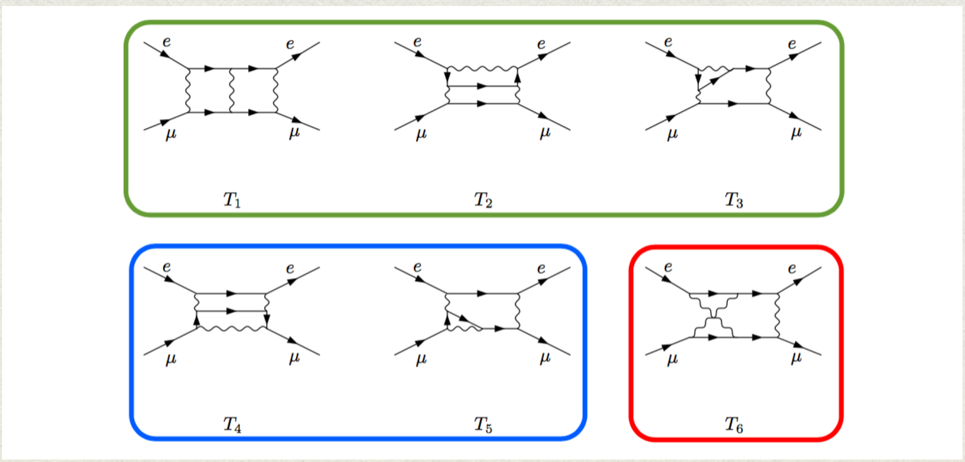
$$\mathcal{M}^{(2)}\left(e\mu \to e\mu\right) = \sum_k c_k\left(s,t,m^2,\epsilon\right) I_k^{(2)}\left(s,t,m^2,\epsilon\right)$$

$O(10000)$ monomials

$O(100)$ MIs

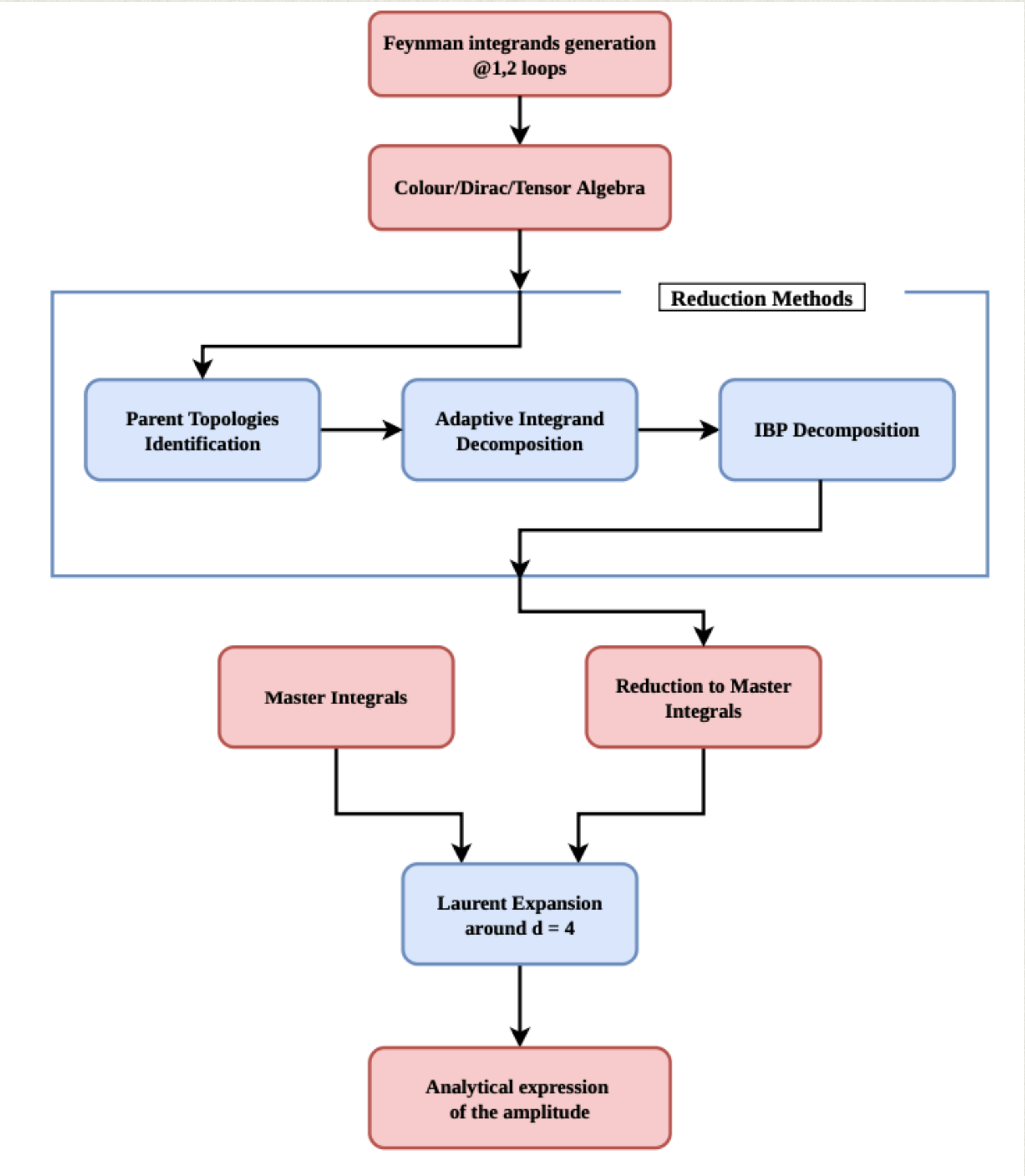
- Evaluate Feynman integrals

[Bonciani, Ferroglia, Gehrmann, von Manteuffel (2008-13)]
[Mastrolia, Passera, Primo, Schubert (2017)]
[Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)]



eμ-scattering @ NNLO

- Let's recap!



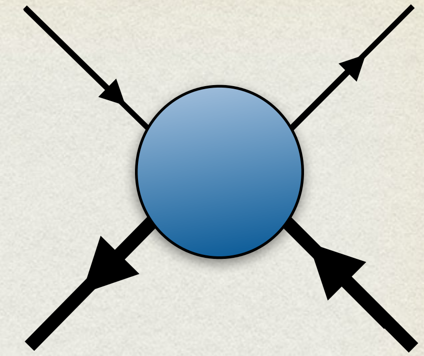
- $e^+e^- \to \mu^+\mu^-$ [Bonciani *et al* (2021)]
- $\mu^+e^- \to \mu^+e^-$ [Broggio *et al* (2022)]
- $q\bar{q} \to t\bar{t}$ [Mandal *et al* (2023)]

<<Yannick's talk

- Public repositories w/ analytic expressions of amplitudes.

[Aida :: Mastrolia, Peraro, Primo, Ronca, W.J.T.]

Numerical evaluation of eμ @ two loops



$$\mathscr{M}^{(2)}_{(2,0)} = \frac{1}{4} \sum_{\text{spins}} 2\text{Re}\left(A_n^{(2)} A_n^{(0)*}\right)$$

$$\begin{aligned}\mathcal{M}^{(1)} &= A^{(1)} + \boldsymbol{n_l}\, B_l^{(1)} + \boldsymbol{n_h}\, C_h^{(1)} \\ \mathcal{M}^{(2)} &= A^{(2)} + \boldsymbol{n_l}\, B_l^{(2)} + \boldsymbol{n_h}\, C_h^{(2)} + \boldsymbol{n_l^2}\, D_l^{(2)} + \boldsymbol{n_h n_l}\, E_{hl}^{(2)} + \boldsymbol{n_h^2}\, F_h^{(2)}\end{aligned}$$

[Ginac :: Vollinga, Weinzierl (2004)]
[HandyG :: Naterop, Signer, Y. Ulrich (2019)]

- Evaluation @ $s/M^2 = 5\,, t/M^2 = -5/4\,, \mu = M.$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ
$\mathcal{M}^{(0)}$	-	-	-	-	$\frac{181}{100}$	-2
$A^{(1)}$	-	-	$-\frac{181}{100}$	1.99877525	22.0079572	-11.7311017
$B_l^{(1)}$	-	-	-	-	-0.069056030	4.94328573
$C_h^{(1)}$	-	-	-	-	-2.24934027	2.54943566
$A^{(2)}$	$\frac{181}{400}$	-0.499387626	-35.4922919	19.4997261	48.8842283	-
$B_l^{(2)}$	-	$-\frac{181}{400}$	0.785712779	-16.1576674	-3.75247701	-
$C_h^{(2)}$	-	-	1.12467013	-9.50785825	-25.8771503	-
$D_l^{(2)}$	-	-	-	-	-3.96845688	-
$E_{hl}^{(2)}$	-	-	-	-	-4.88512563	-
$F_h^{(2)}$	-	-	-	-	-0.158490810	-

IR Structure w/ massive particles in the loop

- tree- & one-loop contributions —> two-loop IR poles after UV renormalisation

[Czakon, Mitov, Moch (2007)]
[Becher, Neubert (2009)]
[Hill (2017)]

$$\mathcal{M}^{(1)}\Big|_{\text{poles}} = \frac{1}{2} Z_1^{\text{IR}} \mathcal{M}^{(0)}\Big|_{\text{poles}}$$

$$\mathcal{M}^{(2)}\Big|_{\text{poles}} = \frac{1}{8}\left[\left(Z_2^{\text{IR}} - \left(Z_1^{\text{IR}}\right)^2\right)\mathcal{M}^{(0)} + 2 Z_1^{\text{IR}} \mathcal{M}^{(1)}\right]\Big|_{\text{poles}}$$

- Anomalous dimension —> IR structure

$$\Gamma = \gamma_{\text{cusp}}(\alpha)\ln\left(-\frac{s}{\mu^2}\right) + 2\gamma_{\text{cusp}}(\alpha)\ln\left(\frac{t-M^2}{u-M^2}\right) + \gamma_{\text{cusp},M}(\alpha,s) + \gamma_h(\alpha,s) + \gamma_l(\alpha,s)$$

- IR renormalisation factor

$$\ln Z^{\text{IR}} = \frac{\alpha}{4\pi}\left(\frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}\right) + \left(\frac{\alpha}{4\pi}\right)^2\left(-\frac{3\beta_0\Gamma_0'}{16\epsilon^3} + \frac{\Gamma_1' - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}\right) + \mathcal{O}\left(\alpha^3\right)$$

$$\Gamma' = \frac{\partial}{\partial \ln\mu}\Gamma(\alpha)$$

☑ Full agreement of the IR poles structure obtained by direct calculation of the two-loop diagrams

real & imaginary part!

PhokharaX towards NNLO

Phokhara_10

[H. Czyż, J.H. Kühn, G. Rodrigo, *et al (2001 — 2020)*]

Current Status

- Monte Carlo event generator for $e^+e^- \to \mu^+\mu^-$/hadrons (+ photons)
- simulates this process at the next-to-leading order (NLO) accuracy.

- Rely :: on QED for pure fermionic processes
  :: on sQED for processes w/ pions

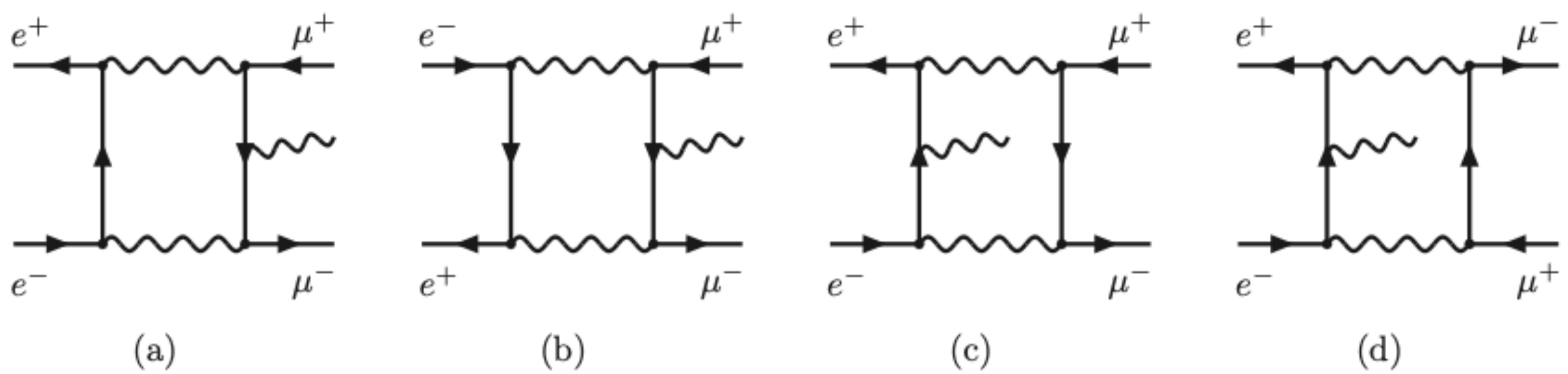


Figure 2.2. The four diagrams with pentagon topology for $e^+e^- \to \mu^+\mu^-\gamma$

- Control on Initial State & Final State radiation

<<Henryk's talk

Phokhara_10

[H. Czyż, J.H. Kühn, G. Rodrigo, *et al (2001 — 2020)*]

Currently in Liverpool

- Revisit structure of Phokhara
- Amplitude generation @ tree and one-loop

- Unify third party Phokhara developments

Olga Shekhovtsova
Stefan Mueller
Fedor Ignatov
...

- Understand experimental needs (Kloe analysis)

Graziano Venanzoni
Paolo Beltrame
Lorenzo Punzi
...

Phokhara_10

Short & long term goals

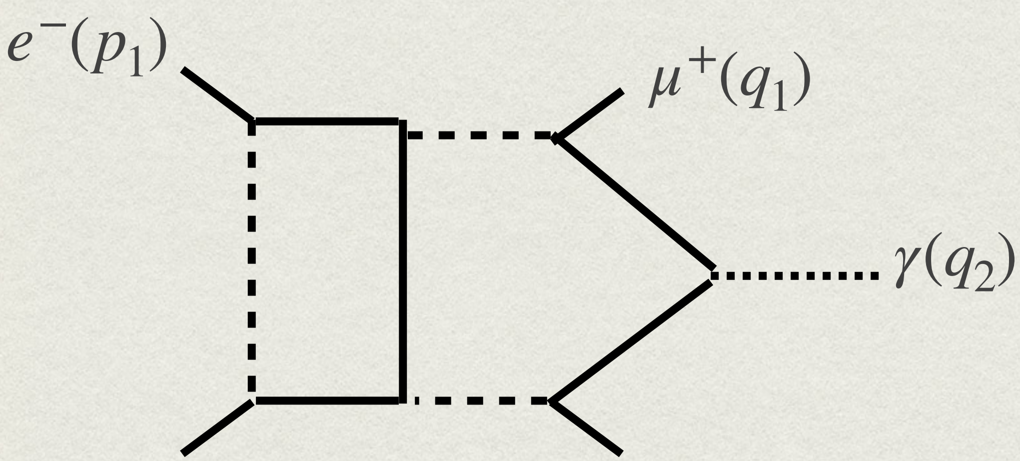
w/ Thomas Teubner & Graziano Venanzoni

- Optimisation of event generators
- Improvement on evaluation of scattering amplitudes

Pau Petit Rosas

- Insertion of scattering amplitudes relevant at NNLO

Tom Dave



Loop integrals known —> Assemble pieces

- Synergy between muon & pion channel
- sQED vs ad-hoc models

Final remark

All technology that is gonna be developed in our team *is not only restricted* to implementations within `Phokhara` framework

Mesmer

McMule

KKMC

Phokhara

The high-energy community has done a lot of work —> Let's use it!

Conclusions & Outlook

We have reached:

- ☑ Complete setup for calculating two-loop scattering amplitudes.
- ☑ NNLO prediction for $e - \mu$ elastic scattering :: MUonE.
- ☑ Baby steps towards Phokhara @ NNLO.

We are working on:

- ☐ Provide efficient approaches to combine numerics & analytics.
- ☐ Use novel mathematic insights in evaluation of Feynman integrals.
- ☐ Extension to further QED processes.
- ☐ Provide a standalone MC event generator.

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