The Hadronic Vacuum Polarization from the lattice

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II Workshop on Muon Precision Physics

Liverpool

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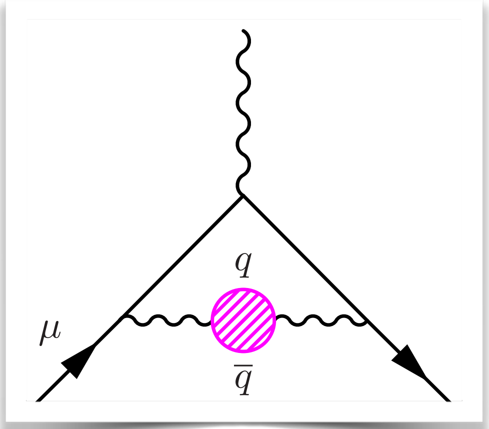
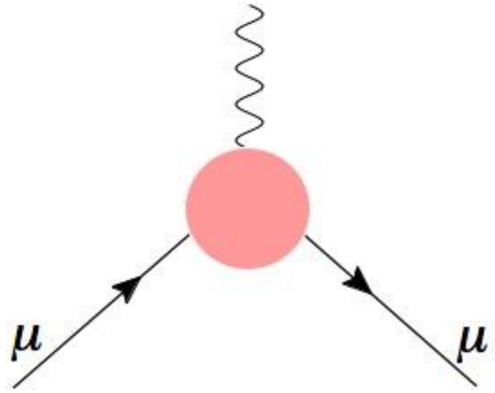
OUTLINE

- Introduction
- HVP from the lattice
- Window observables
- Further connections



Introduction

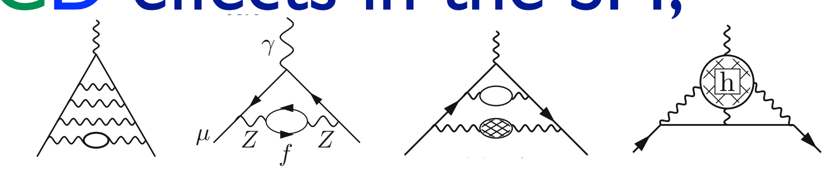
Muon magnetic anomaly



$$= (-i\,e)\,\bar{u}(p')\left[\gamma^{\mu}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2)\right]u(p)$$

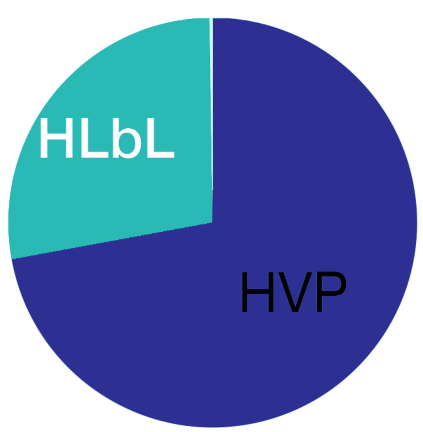
<u>muon anomalous magnetic moment:</u> $a_\mu \equiv \frac{g_\mu - 2}{2} = F_2(0)$

- is generated by quantum loops;
- receives contribution from QED, EW and QCD effects in the SM;
- is a sensitive probe of new physics

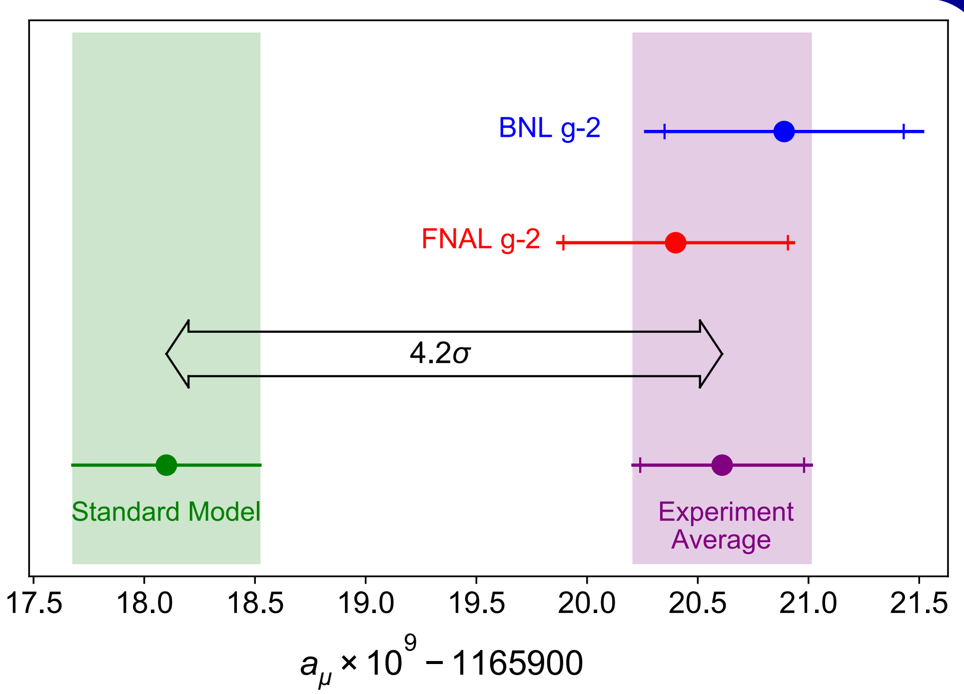


SM contributions to $a_\mu [\times 10^{10}]$

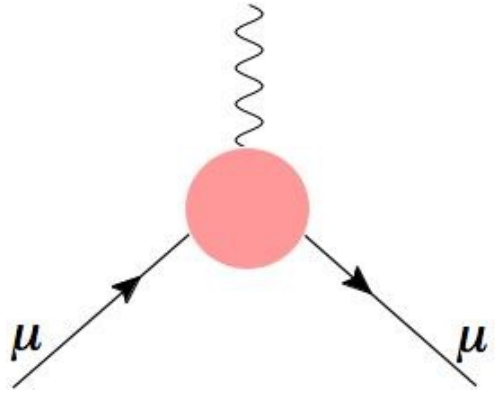
5-loop QED	11 658 471.8931(104)
2-loop EW	15.36(10)
HVP LO	693.1(4.0)
HVP NLO	-9.83(7)
HVP NNLO	1.24(1)
HLbL	9.2(1.8)



Theory error dominated by hadronic physics



Muon magnetic anomaly



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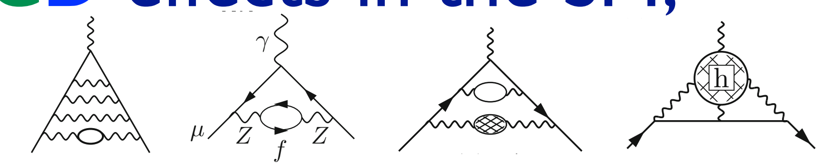

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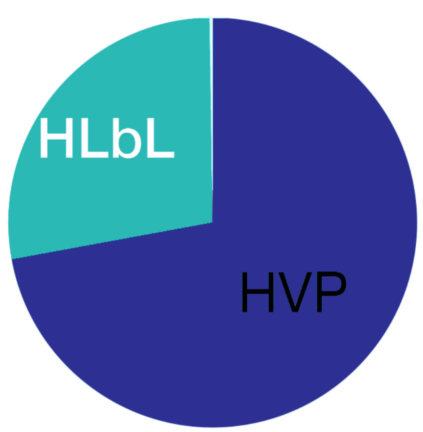
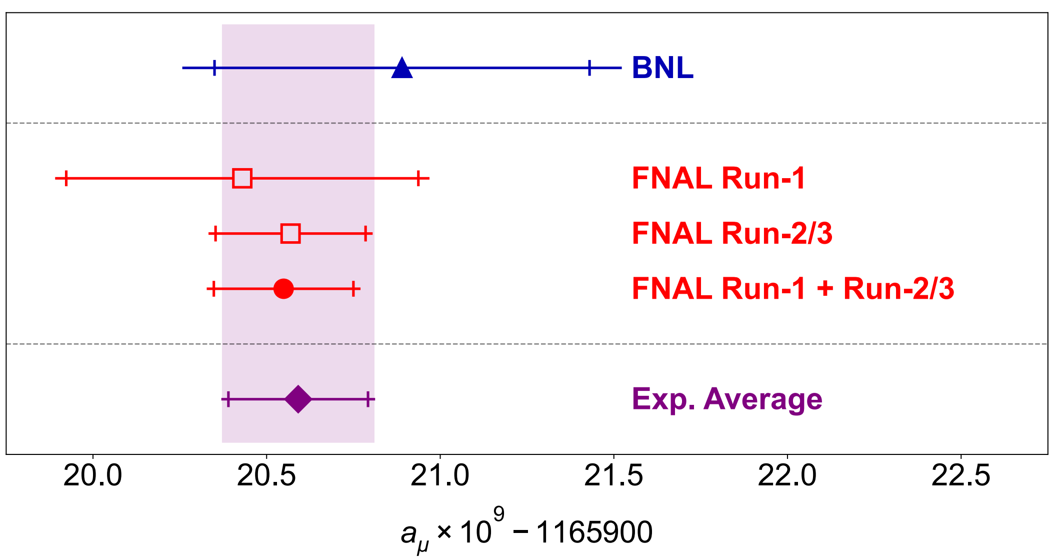




Muon g-2 2023

Theory error dominated by hadronic physics

Precision goal for Fermilab ×4 better implies knowing HVP at 0.2-0.3% accuracy

Hadronic contributions

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 718.9(4.1) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{had}}$$

Clearly right order of magnitude:

$$a_\mu^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

(already Gourdin & de Rafael '69 found $a_\mu^{\text{had}} = 650(50) \times 10^{-10}$)

Huge challenge: theory of strong interaction between quarks and gluons, QCD, hugely nonlinear at energies relevant for a_μ

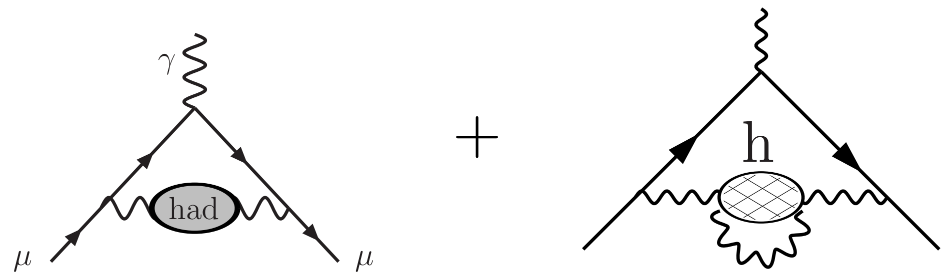
$\rightarrow$ perturbative methods used for electromagnetic and weak interactions do not work

$\rightarrow$ need nonperturbative approaches

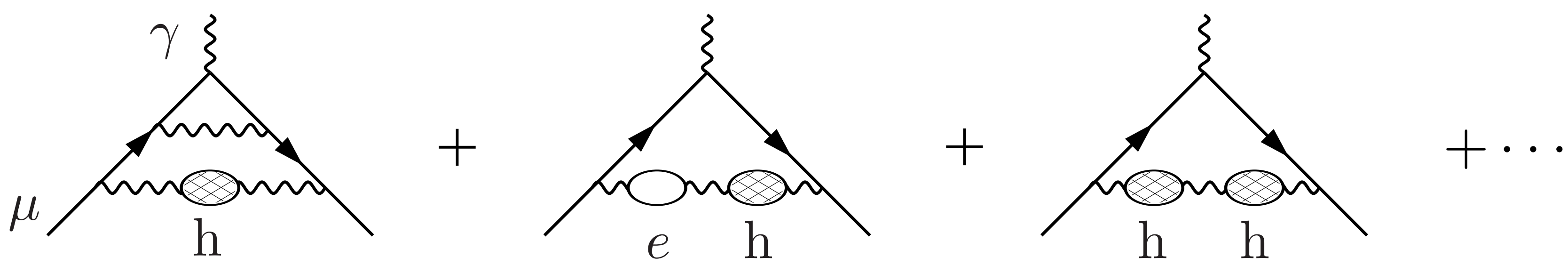
Write

$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$

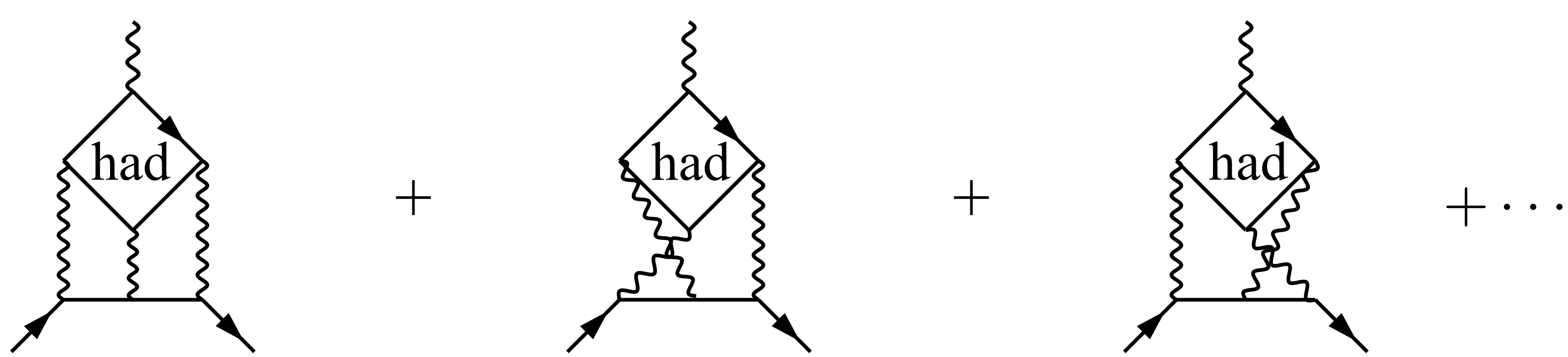
Hadronic contributions: diagrams



$$\to a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$$

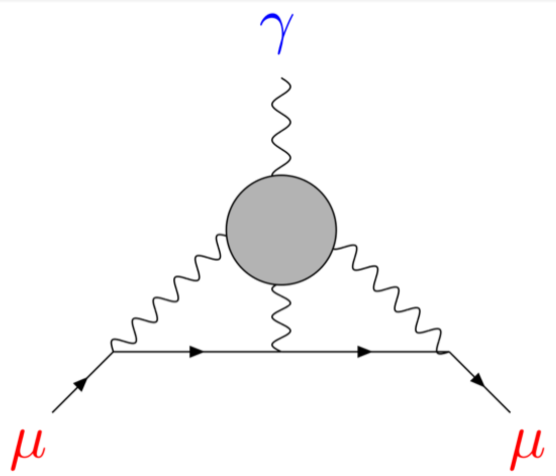


$$\to a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

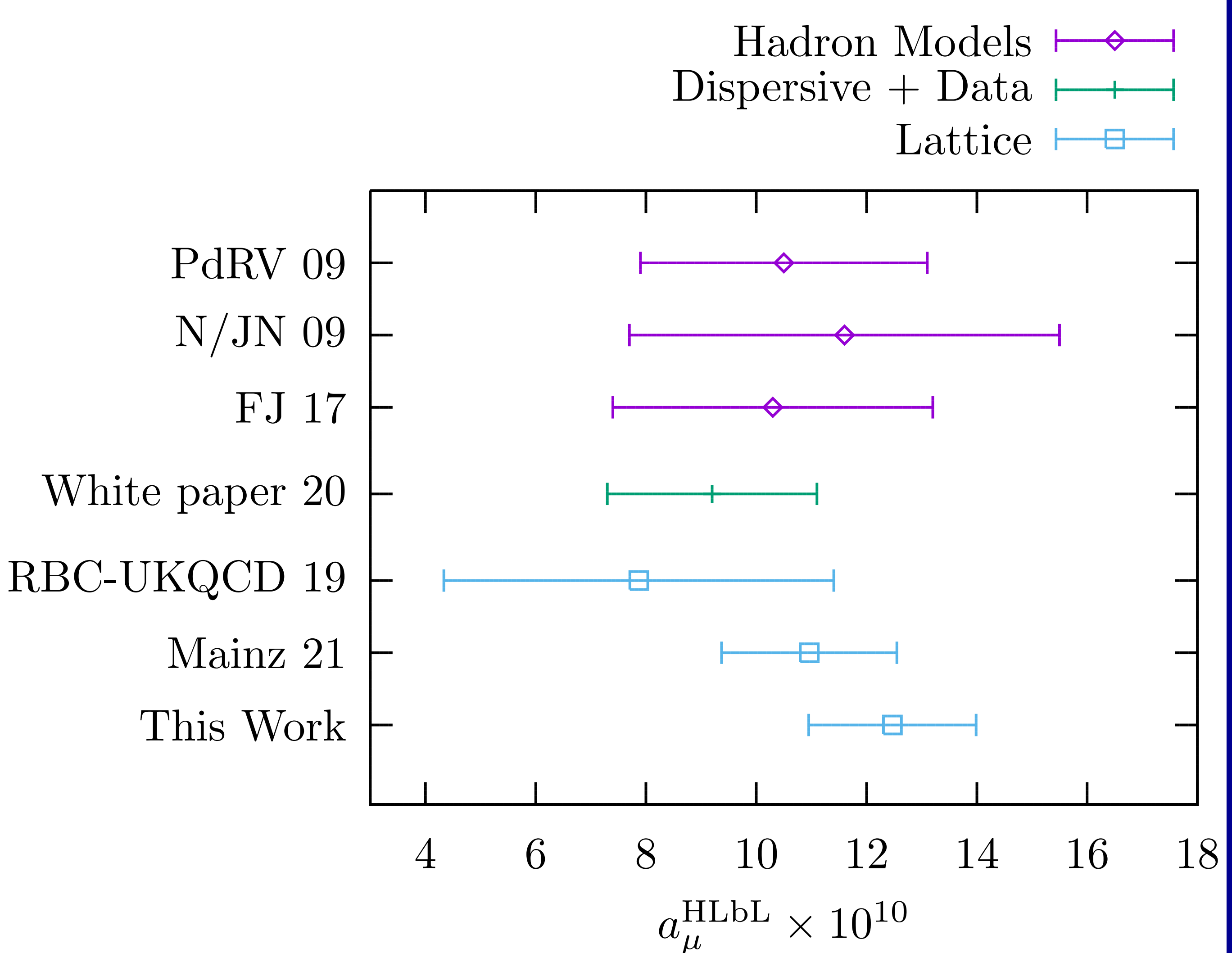


$$\to a_\mu^{\text{HLbL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

Hadronic light-by-light



- HLbL much more complicated than HVP, but ultimate precision needed is $\simeq$ 10% instead of $\simeq$ 0.2%
- For many years, only accessible to models of QCD w/ difficult to estimate systematics (Prades et al '09):
  $a_\mu^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$
- Also, lattice QCD calculations were exploratory and incomplete
- Tremendous progress in past 5 years:
  - $\rightarrow$ Phenomenology: rigorous data driven approach [Colangelo, Hoferichter, Kubis, Procura, Stoffer,... '15-'20]
  - $\rightarrow$ Lattice: first two solid lattice calculations
- All agree w/ older model results but error estimate much more solid and will improve
- Agreed upon average w/ NLO HLbL and conservative error estimates [WP '20]
- $a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} - a_\mu^{\text{HLbL}} = 709.7(4.5) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{HVP}}$



[Blum et al '23]

Standard Model prediction vs Experiment

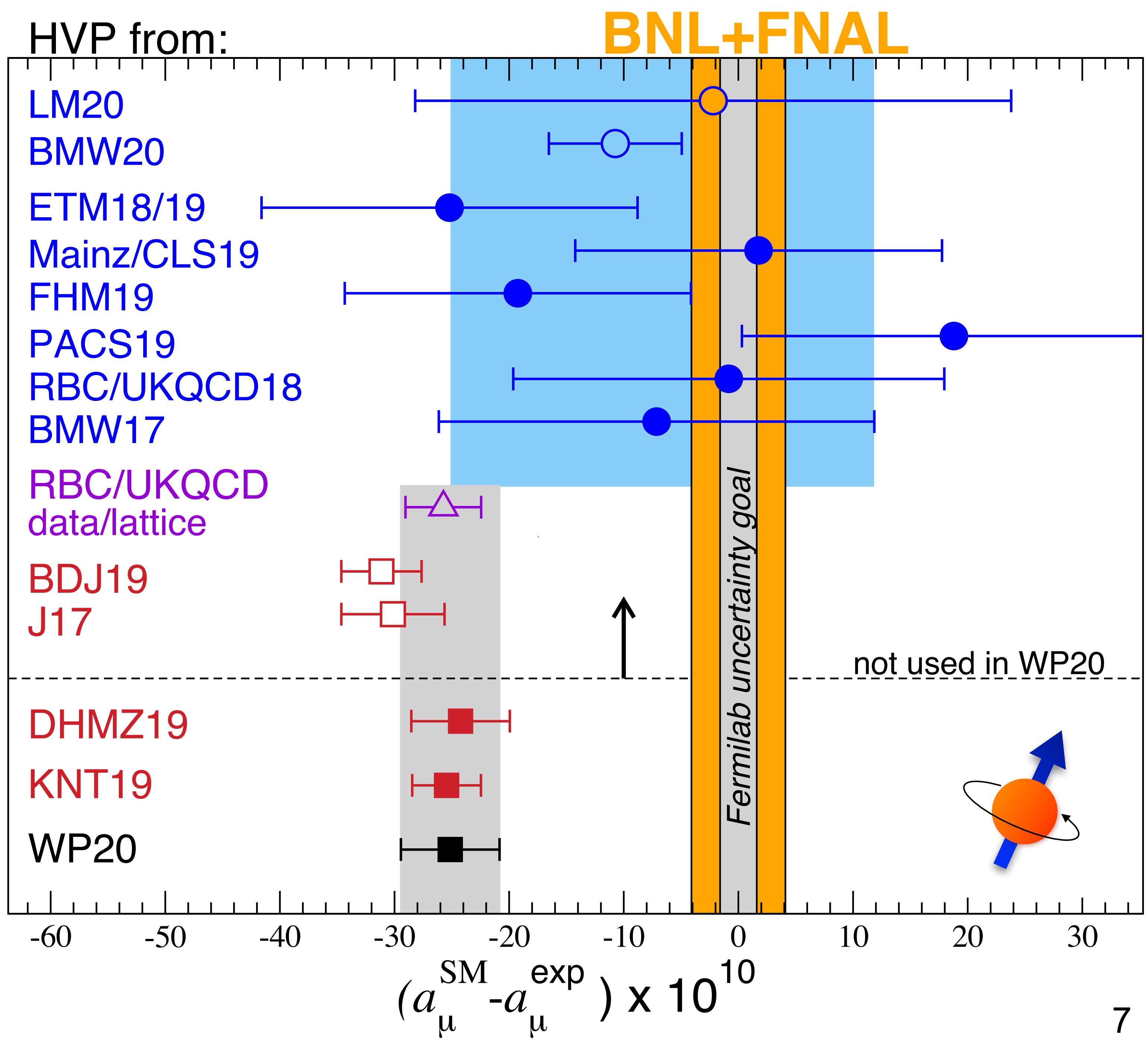
$$a_\mu^{\mathrm{SM}} \;\; a_\mu^{\mathrm{HVP}} + \left[a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{Weak}} + a_\mu^{\mathrm{HLbL}}\right]$$

Lattice QCD + QED

hybrid: combine data & lattice

data driven

+ unitarity/analyticity constraints

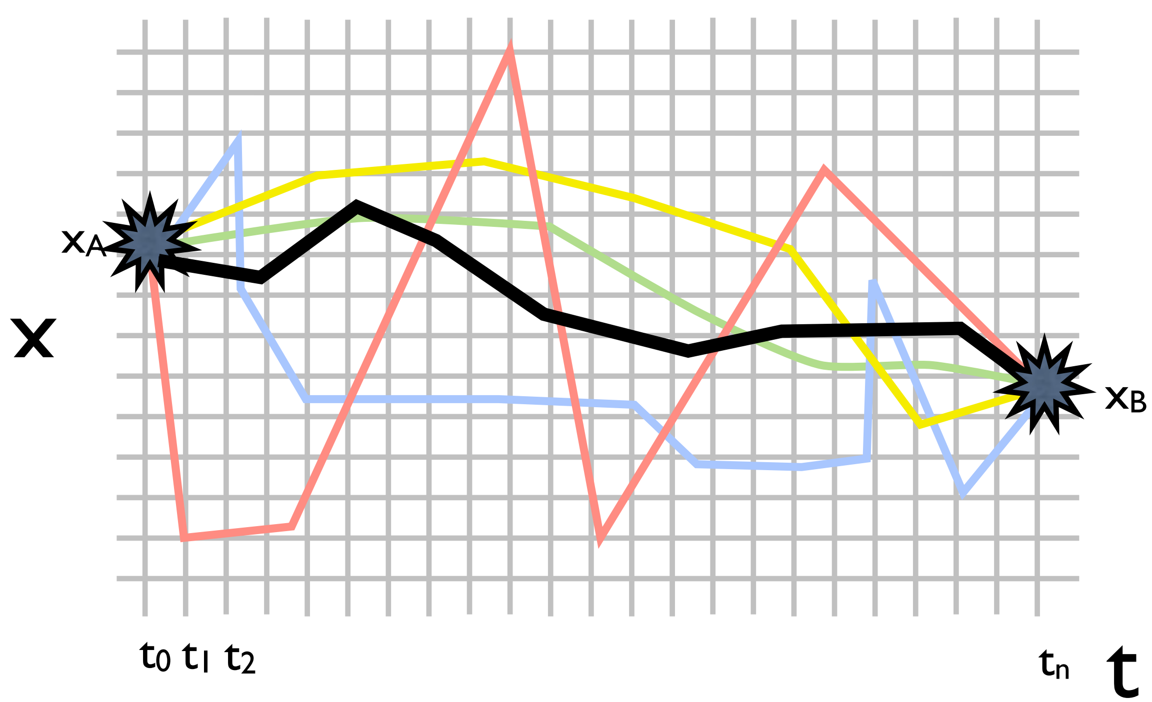


Small interlude:
Lattice QCD

Lattice QCD

Numerical first-principles approach to non-perturbative QCD

- Discretise QCD onto 4D space-time lattice
- QCD equations ⟷ integrals over the values of quark and gluon fields on each site/link (QCD path integral)
- ~10^{12} variables (for state-of-the-art)

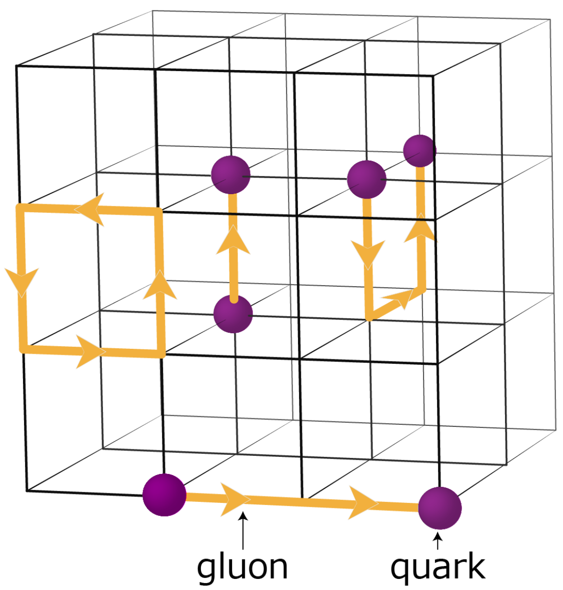


- Evaluate by importance sampling
- Paths near classical action dominate
- Calculate physics on a set (ensemble) of samples of the quark and gluon fields

Lattice QCD

Numerical first-principles approach to
non-perturbative QCD

- Euclidean space-time $t \to i\tau$
- Finite lattice spacing a
- Volume $L^3 \times T = 64^3 \times 128$
- Boundary conditions

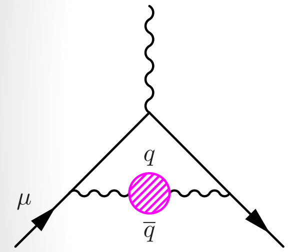


Approximate the QCD path integral by **Monte Carlo**

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\overline{\psi} \mathcal{D}\psi \mathcal{O}[A, \overline{\psi}\psi] e^{-S[A,\overline{\psi}\psi]} \Rightarrow \langle \mathcal{O} \rangle \simeq \frac{1}{N_{\mathrm{conf}}} \sum_{i}^{N_{\mathrm{conf}}} \mathcal{O}([U^i])$$

with field configurations U^i distributed according to $e^{-S[U]}$

HVP from the lattice

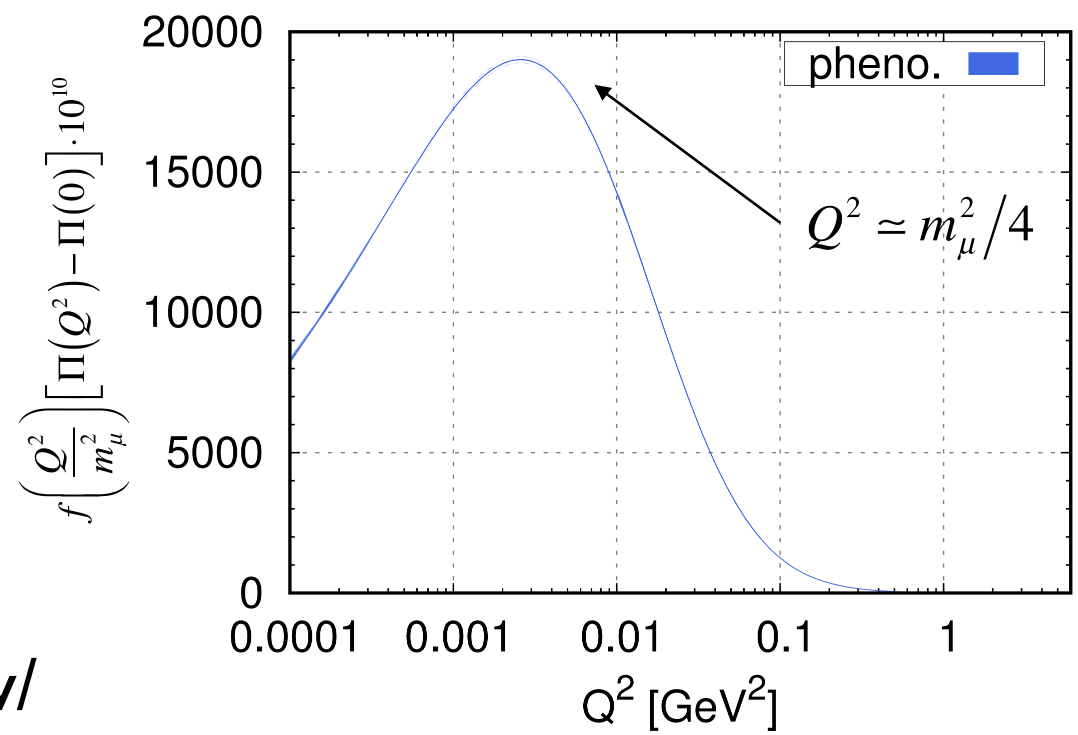


HVP from LQCD

$$\Pi_{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot x} \left\langle J_\mu(x) J_\nu(0) \right\rangle = \left[\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu\right] \Pi\left(Q^2\right)$$

$$a_\mu^{\rm HVP,LO} = 4\alpha_{em}^2 \int_0^\infty dQ^2 \, \frac{1}{m_\mu^2} f\left(\frac{Q^2}{m_\mu^2}\right) \left[\Pi\left(Q^2\right) - \Pi(0)\right]$$

B. E. Lautrup *et al.*, 1972



F. Jegerlehner, "alphaQEDc17"

FV & $a \neq 0$: A. discrete momenta ($Q_{\rm min} = 2\pi/T > m_\mu/2$); B. $\Pi_{\mu\nu}(0) \neq 0$ in FV contaminates $\Pi(Q^2) \sim \Pi_{\mu\nu}(Q)/Q^2$ for $Q^2 \to 0$ w/ very large FV effects; C. $\Pi(0) \sim \ln(a)$

↓

Time-Momentum Representation

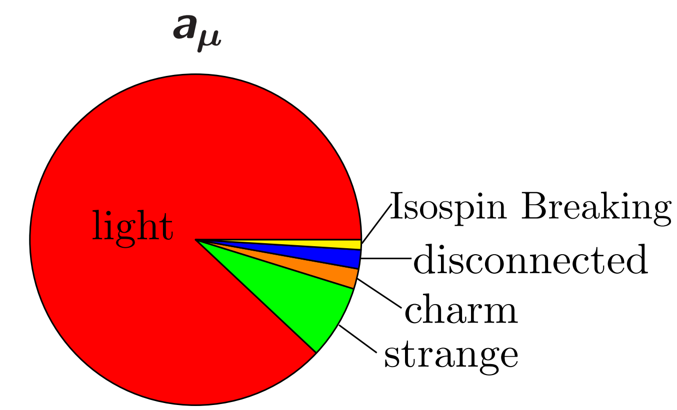
$$a_\mu^{\rm HVP,LO} = 4\alpha_{em}^2 \int_0^\infty dt\; \tilde{f}(t)\, V(t)$$

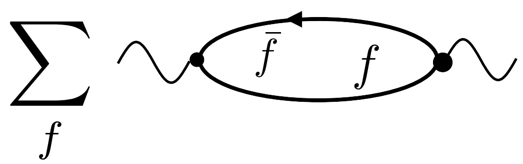
$$V(t) \equiv \frac{1}{3} \sum_{i=1,2,3} \int d\vec{x} \left\langle J_i(\vec{x},t) J_i(0) \right\rangle$$

D. Bernecker and H. B. Meyer, 2011

Time-Momentum Representation

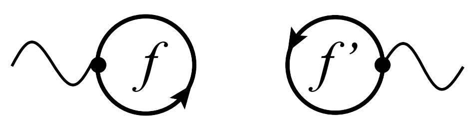
- No reliance on exp. data, except for hadronic quantities used to calibrate the simulation ($M_\pi, M_K, M_{nucl}, \ldots$)
- Can perform an explicit quark flavor separation of $a_\mu^{\text{HVP,LO}}$



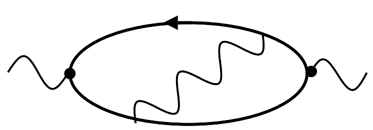


light-quark connected $a_\mu^{\text{HVP,LO}}(\text{ud}) \sim 90\,\%$ of total

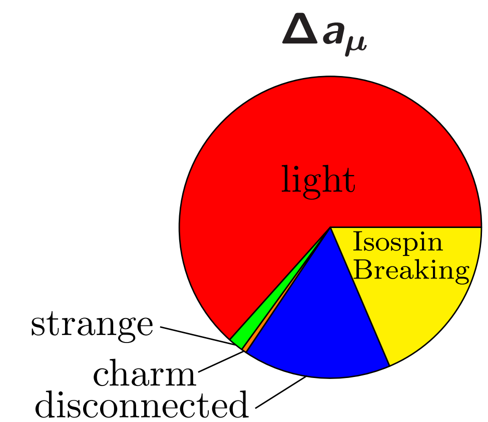
s,c-quark connected $a_\mu^{\text{HVP,LO}}(\text{s, c}) \sim 8\,\%\,, 2\,\%$ of total



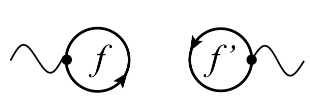
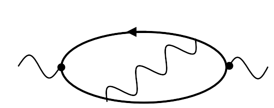
disconnected $a_{\mu,disc}^{\text{HVP,LO}} \sim 2\,\%$ of total

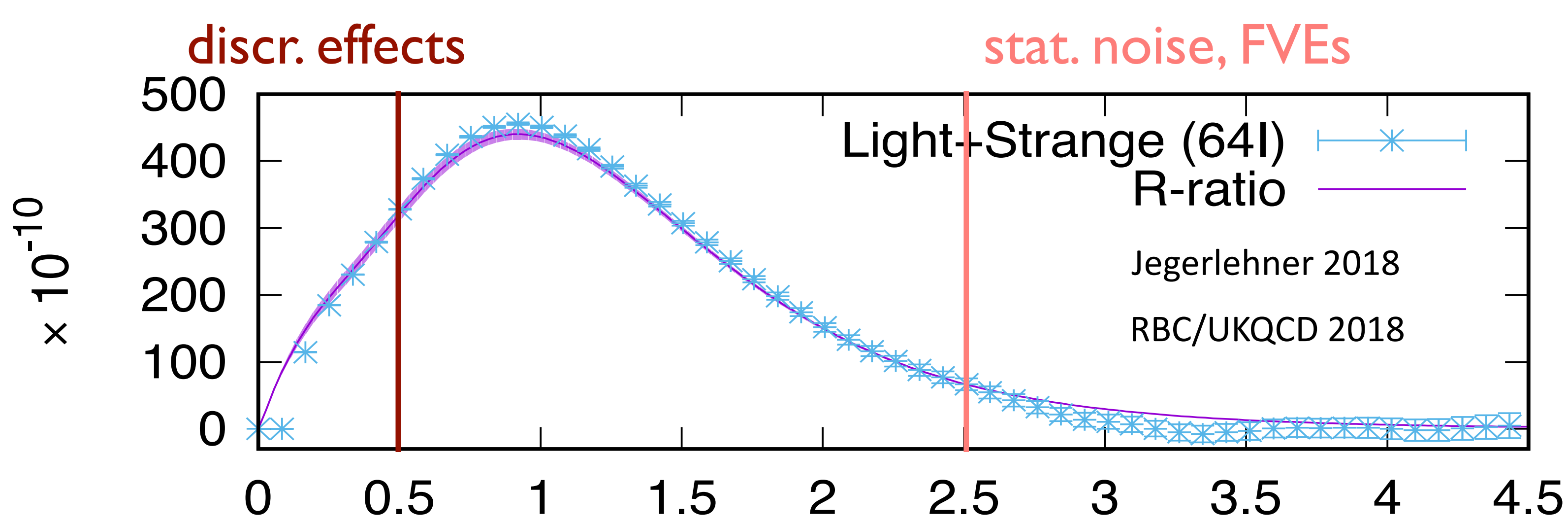


IB ($m_u \neq m_d + \text{QED}$) $\delta a_\mu^{\text{HVP,LO}} \sim 1\,\%$ of total



Challenges:

- sub-percent stat. precision
  exp. growing StN ratio in $V(t)$ as $t \to \infty$
- correct for FVEs, control discr. effects (scale setting and continuum extrap.)
- quark-disconn. diagrams 
  control stat. & stochastic noise
- isospin-breaking: $m_u \neq m_d$, $\alpha_{em} \neq 0$ 

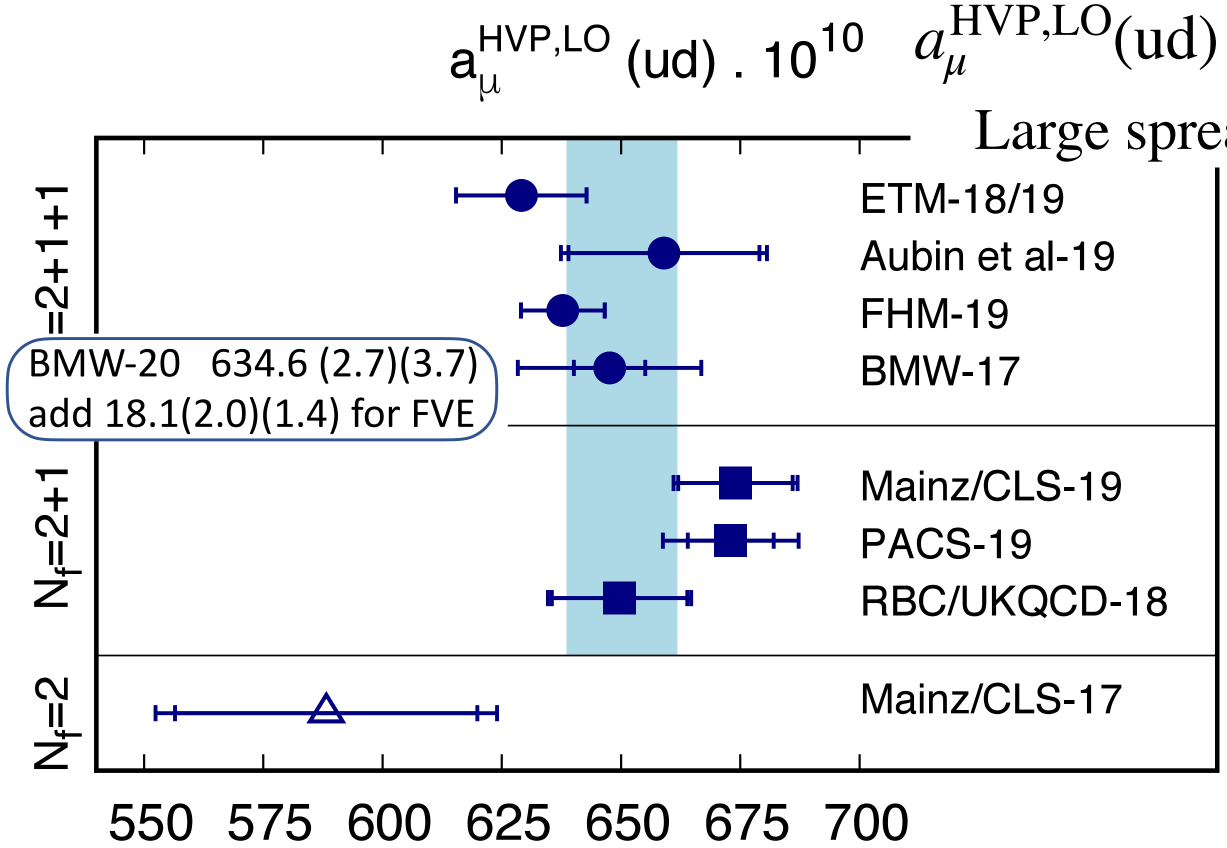


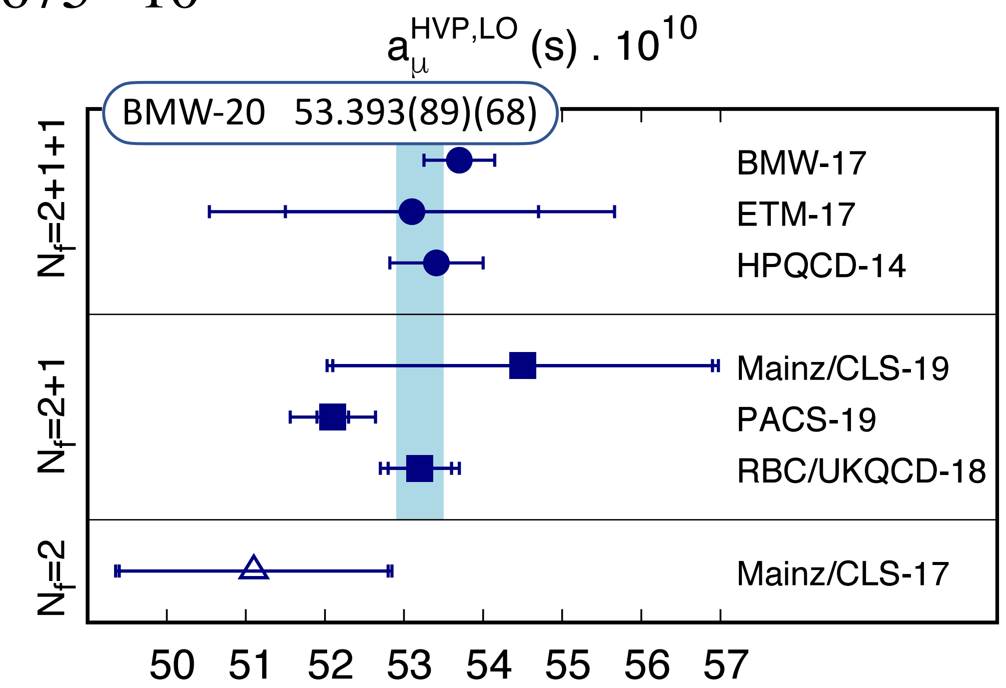
Results for each contribution

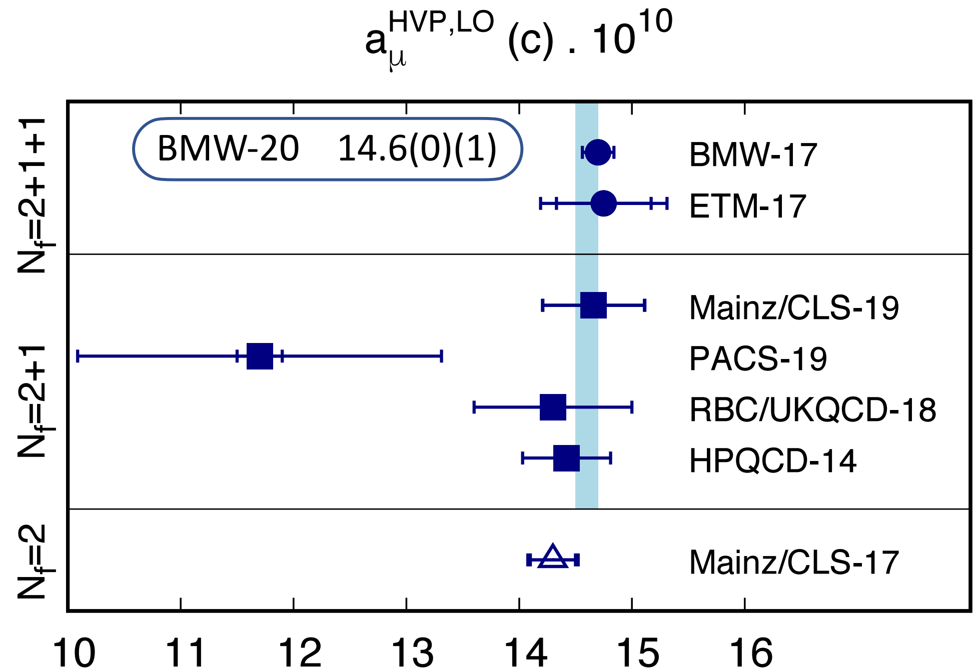
$$a_\mu^{\mathrm{HVP,LO}}(\mathrm{ud}) = 650.2(11.6) \cdot 10^{-10}$$

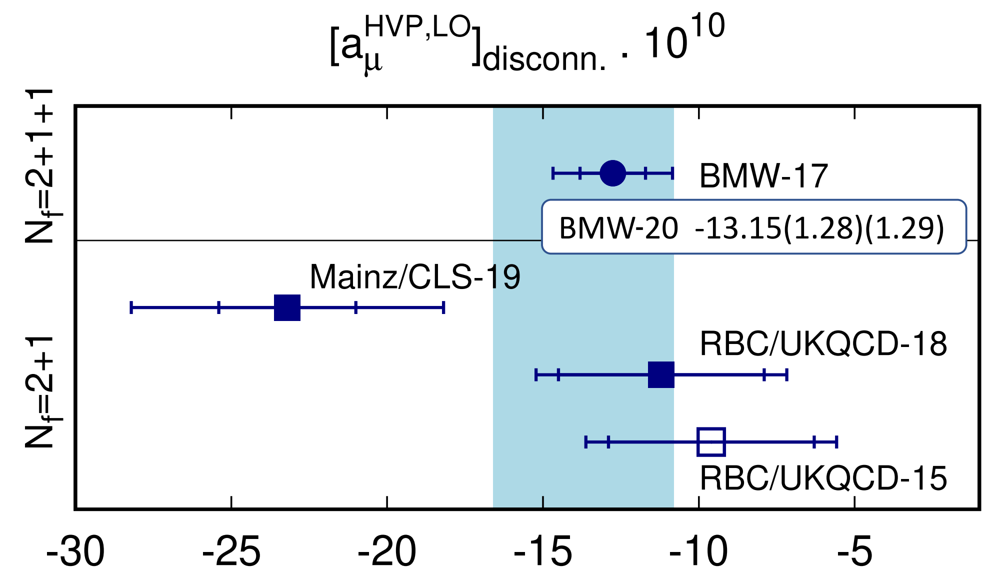
Large spread $630 \div 675 \cdot 10^{-10}$

WP '20

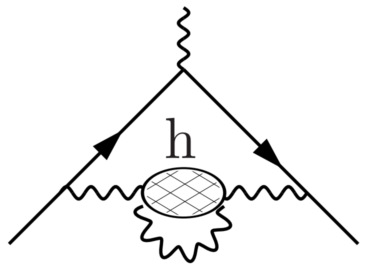


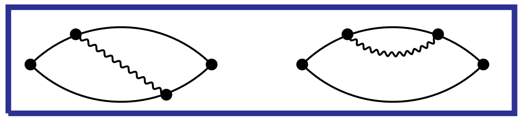




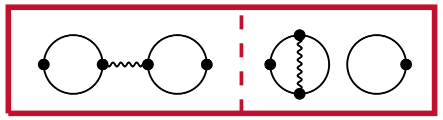


Isospin-breaking contributions



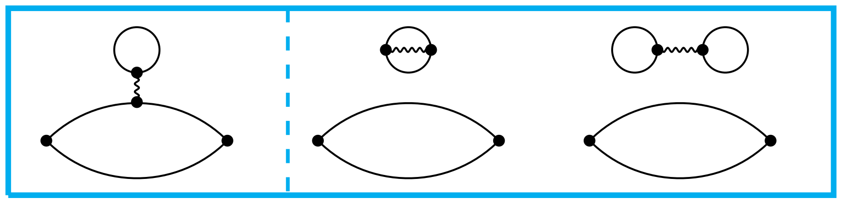


BMW	$-1.27(40)(33)$
RBC/UKQCD	$5.9(5.7)(1.7)$
ETM	$1.1(1.0)$

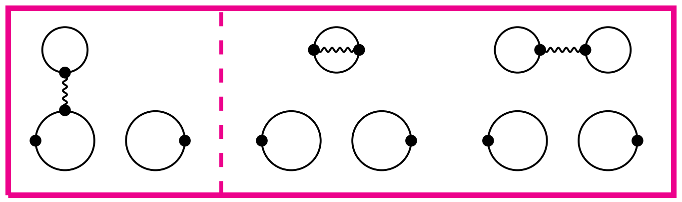


$-0.55(15)(11)$ BMW

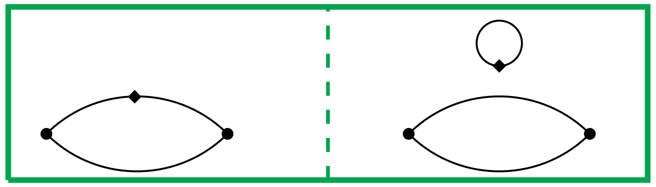
$-6.9(2.1)(2.0)$ RBC/UKQCD



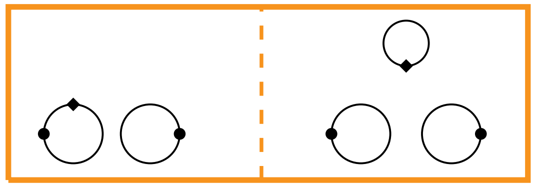
$-0.0095(86)(99)$ $0.42(20)(19)$ BMW



$0.011(24)(14)$ $-0.047(33)(23)$ BMW

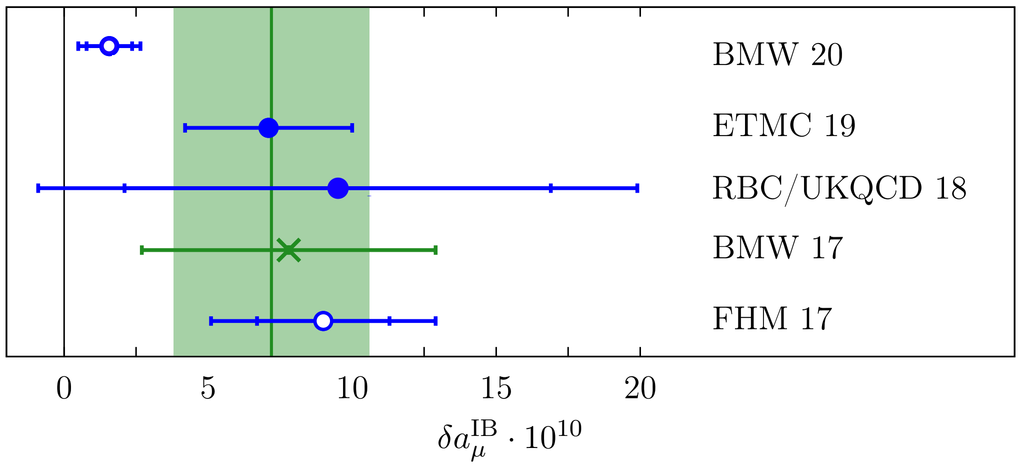


$6.59(63)(53)$		BMW
$10.6(4.3)(6.8)$		RBC/UKQCD
$6.0(2.3)$		ETM
$7.7(3.7)$	$9.0(2.3)$	FHM
$9.0(0.8)(1.2)$		LM



$-4.63(54)(69)$ BMW

BMW [arXiv:2002.12347]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]



- Small overall value due to large cancellations
- Large statistical uncertainties
- More precise calculations are in progress

Plots by V. Gülpers and H. Wittig

Window observables

Windows "on the g-2 mystery"

Restrict integration over Euclidean time to sub-intervals

→ reduce/enhance sensitivity to systematic effects

$$a_\mu^{\mathrm{HVP,LO}} = a_\mu^{SD} + a_\mu^{W} + a_\mu^{LD}$$

$$a_\mu^{SD}(f; t_0, \Delta) \equiv 4\alpha_{em}^2 \int_0^\infty dt\, \tilde{f}(t) V^f(t) \left[1 - \Theta\left(t, t_0, \Delta\right)\right]$$

$$a_\mu^{W}(f; t_0, t_1, \Delta) \equiv 4\alpha_{em}^2 \int_0^\infty dt\, \tilde{f}(t) V^f(t) \left[\Theta\left(t, t_0, \Delta\right) - \Theta\left(t, t_1, \Delta\right)\right]$$

$$a_\mu^{LD}(f; t_1, \Delta) \equiv 4\alpha_{em}^2 \int_0^\infty dt\, \tilde{f}(t) V^f(t)\, \Theta\left(t, t_1, \Delta\right)$$

$$\Theta\left(t, t', \Delta\right) = \frac{1}{1 + e^{-2(t-t')/\Delta}}$$

"Standard" choice:

$$t_0 = 0.4 \text{ fm} \quad t_1 = 1.0 \text{ fm}$$

$$\Delta = 0.15 \text{ fm}$$

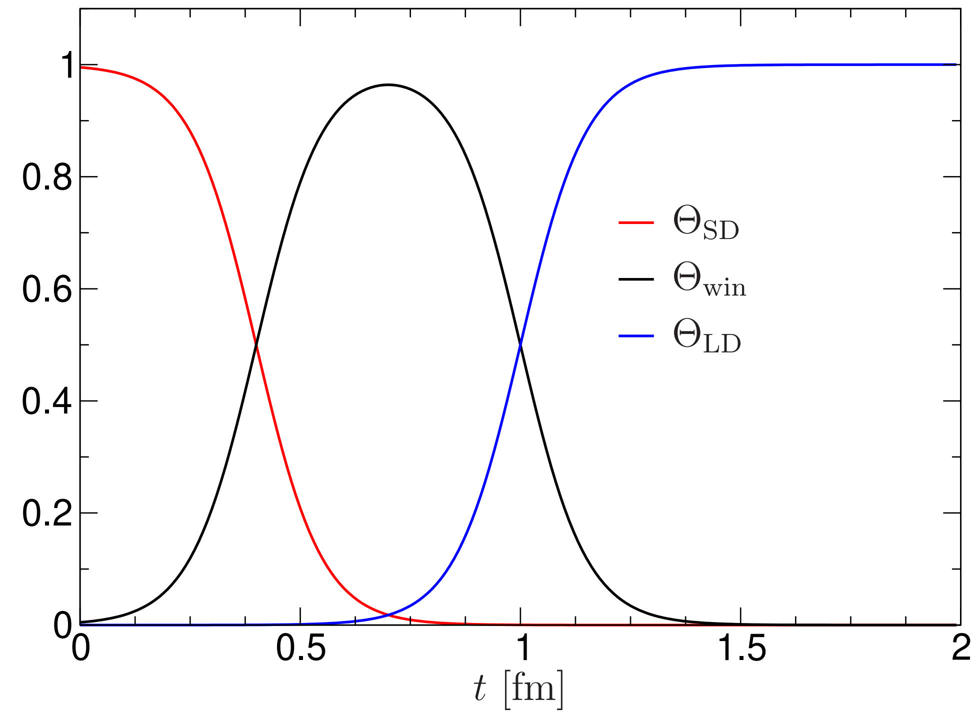
RBC/UKQCD 2018

<u>Intermediate window</u>

- Reduced FVEs
- Much better StN ratio

→ Precision test of different lattice calculations

→ Commensurate uncertainties compared to dispersive evaluations

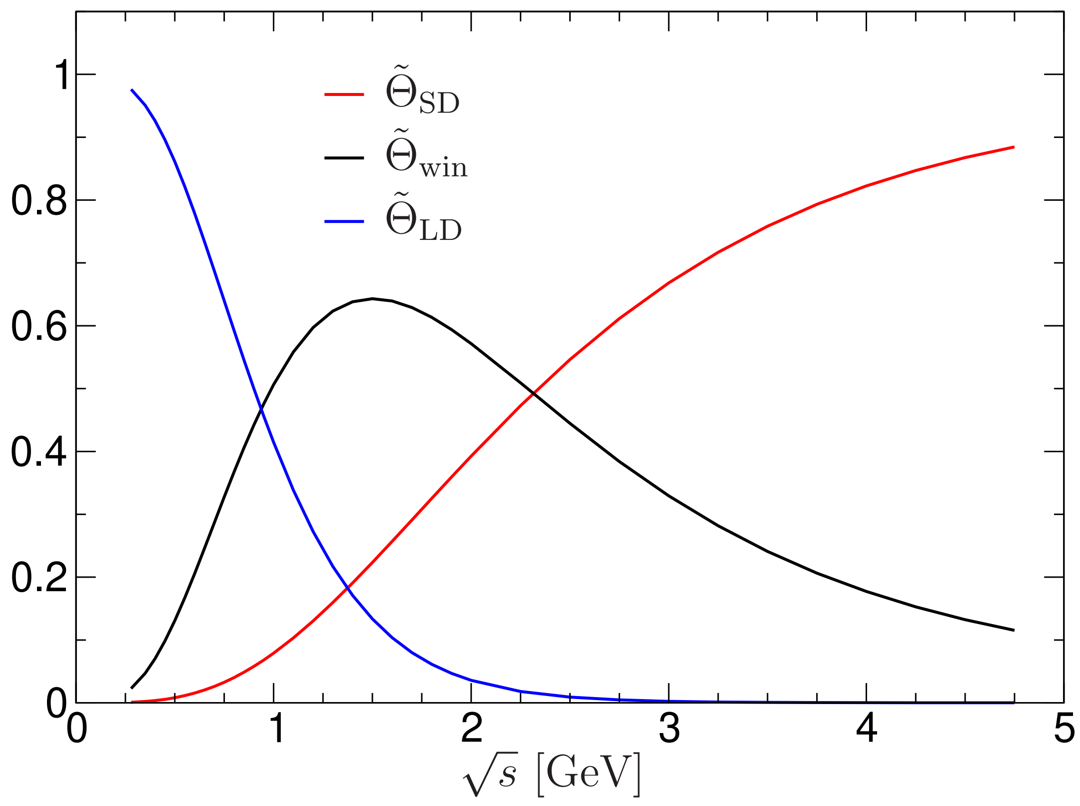


Comparison with R-ratio

$$V(t) = \frac{1}{12\pi^2}\int_{M_{\pi^0}}^{\infty} d(\sqrt{s})\, R(s)\, s\, e^{-\sqrt{s}t} \qquad R(s) = \frac{3s}{4\pi\alpha_{em}^2}\sigma(s, e^+e^- \to hadrons)$$

Insert $V(t)$ into the expression for TMR

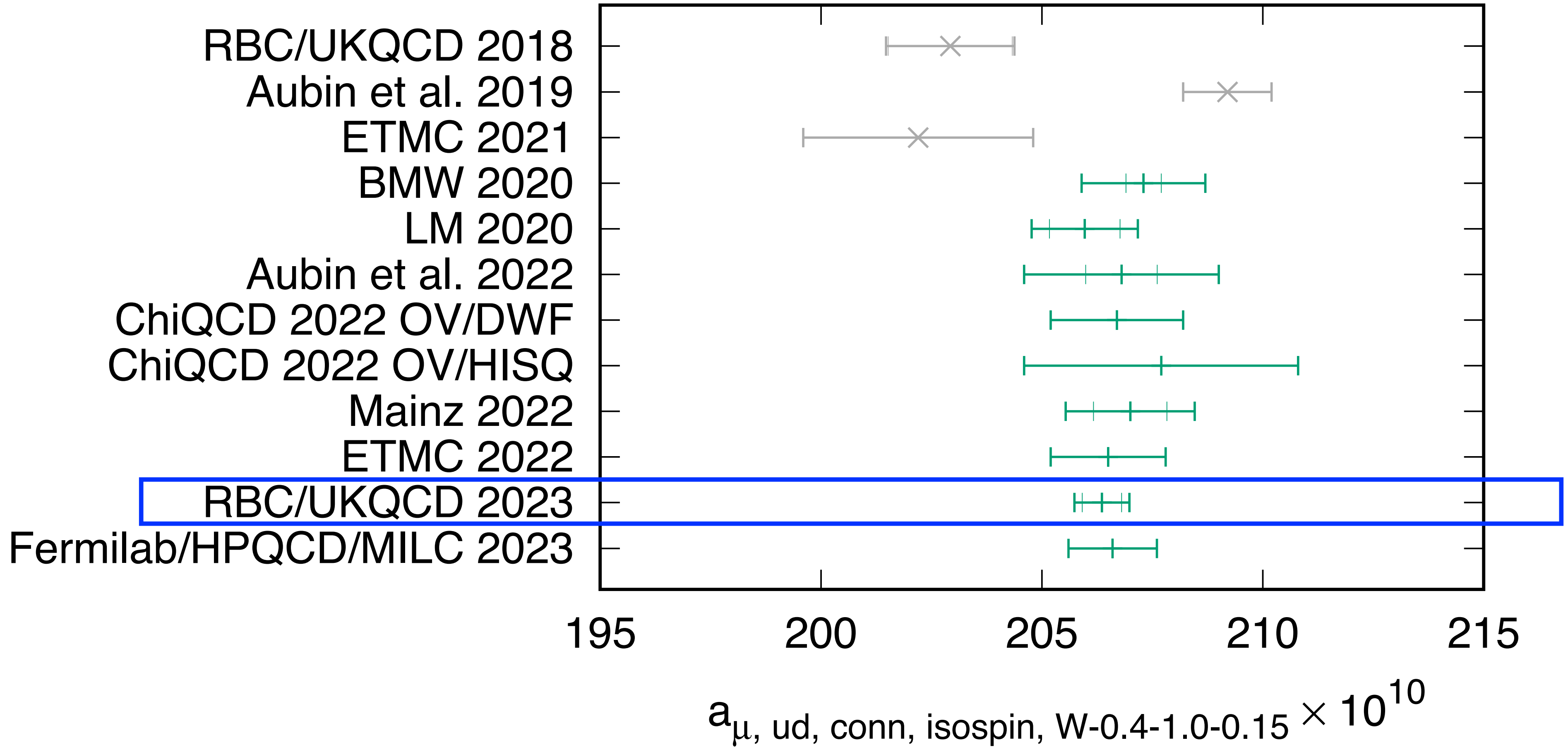
$$a_{\mu,win}^{\mathrm{HVP,LO}} = 4\alpha_{em}^2 \int_{M_{\pi^0}}^{\infty} d(\sqrt{s})R(s)\, \frac{1}{12\pi^2}\, s \int_0^{\infty} dt\, \tilde{f}(t)\, \Theta_{win}(t)\, e^{-\sqrt{s}t}$$



Colangelo *et al.* 2022

	$a_{\mathrm{SD}}^{\mathrm{HVP}}$	$a_{\mathrm{int}}^{\mathrm{HVP}}$	$a_{\mathrm{LD}}^{\mathrm{HVP}}$	$a_{\mathrm{total}}^{\mathrm{HVP}}$
All channels	68.4(5)	229.4(1.4)	395.1(2.4)	693.0(3.9)
	[9.9%]	[33.1%]	[57.0%]	[100%]
2π below 1.0 GeV	13.7(1)	138.3(1.2)	342.3(2.3)	494.3(3.6)
	[2.8%]	[28.0%]	[69.2%]	[100%]
3π below 1.8 GeV	2.5(1)	18.5(4)	25.3(6)	46.4(1.0)
	[5.5%]	[39.9%]	[54.6%]	[100%]
White Paper [1]	–	–	–	693.1(4.0)
RBC/UKQCD [24]	–	231.9(1.5)	–	715.4(18.7)
BMWc [36]	–	236.7(1.4)	–	707.5(5.5)
BMWc/KNT [7, 36]	–	229.7(1.3)	–	–
Mainz/CLS [99]	–	237.30(1.46)	–	–
ETMC [100]	69.33(29)	235.0(1.1)	–	–

Results for the intermediate window



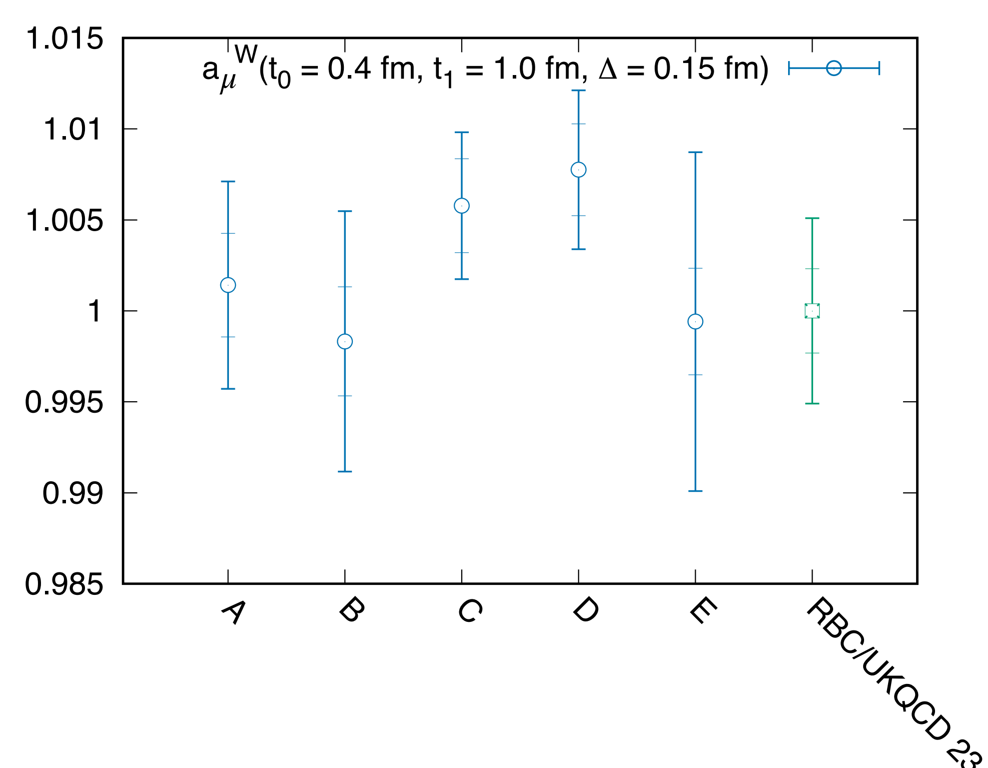
Blinding

- 2 analysis groups for ensemble parameters (not blinded)
- 5 analysis groups for vector-vector correlators (blinded, to avoid bias towards other lattice/R-ratio results)
- Blinded vector correlator $C_b(t)$ relates to true correlator $C_0(t)$ by

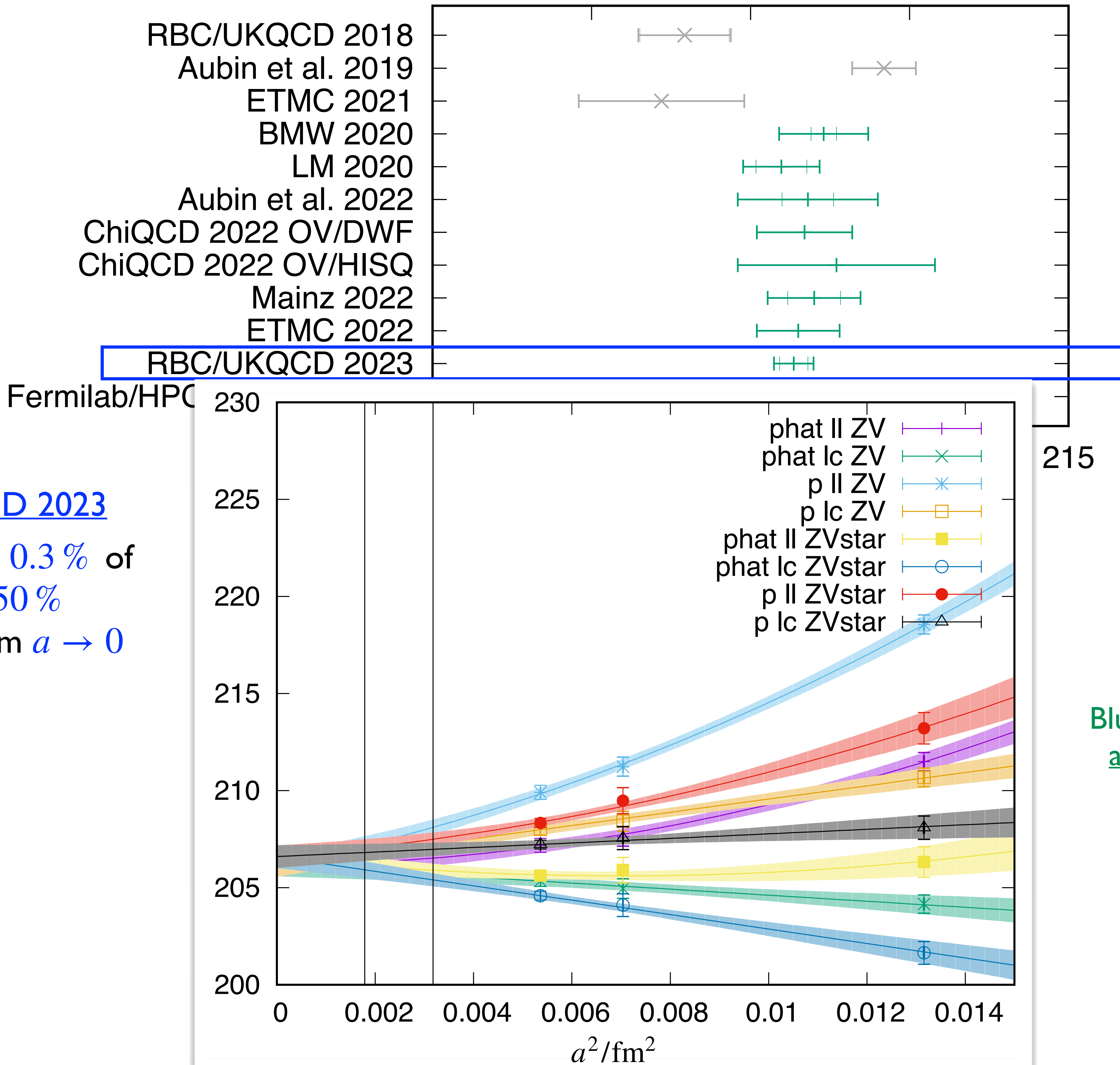
$$C_b(t) = (b_0 + b_1 a^2 + b_2 a^4) C_0(t) \tag{1}$$

with appropriate random b_0, b_1, b_2, different for each analysis group. This prevents complete unblinding based on previously shared data on coarser ensembles.

Relative unblinding (standard window)



Results for the intermediate window

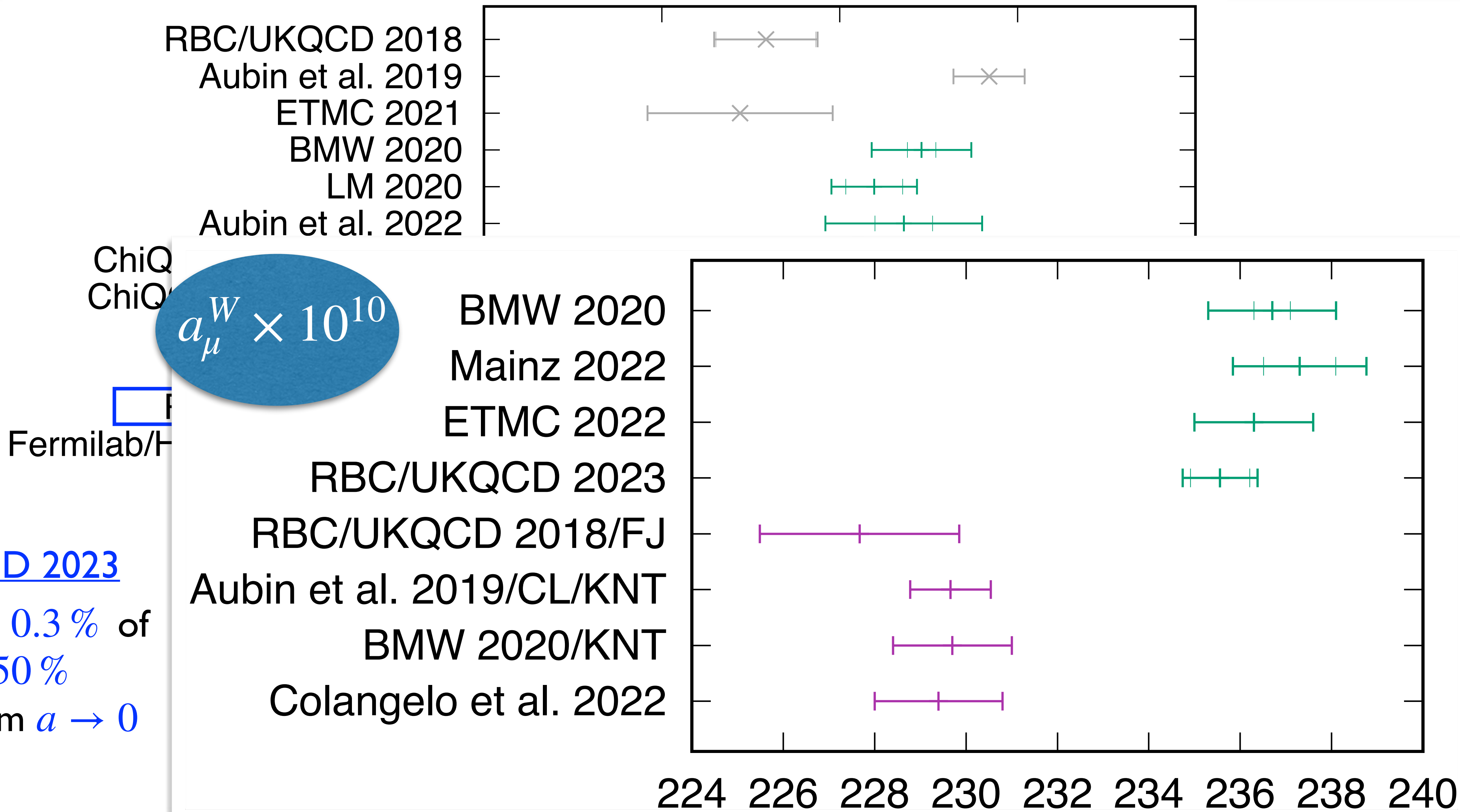


RBC/UKQCD 2023

- tot. err. $\sim 0.3\,\%$ of which $\sim 50\,\%$ comes from $a \to 0$

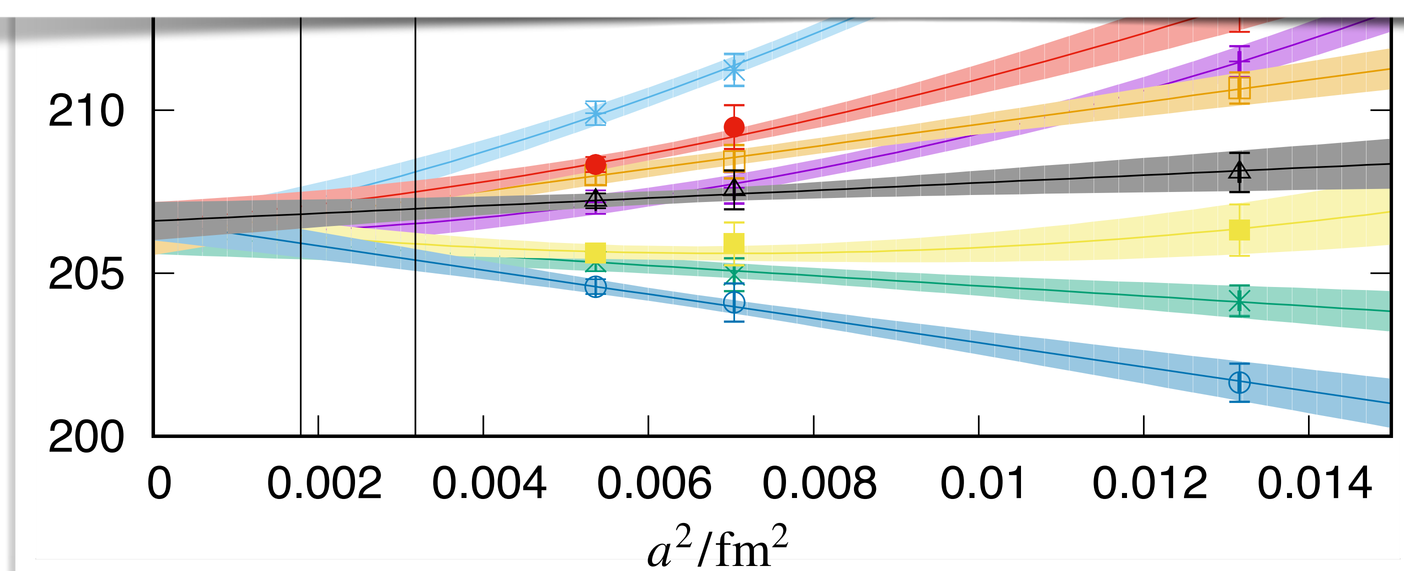
Blum, DG *et al.* 2023
arXiv:2301.08696

Results for the intermediate window



[RBC/UKQCD 2023](#)

- tot. err. $\sim 0.3\,\%$ of which $\sim 50\,\%$ comes from $a \to 0$
- 3.9σ tension w/ Colangelo *et al.* 22/ Lat

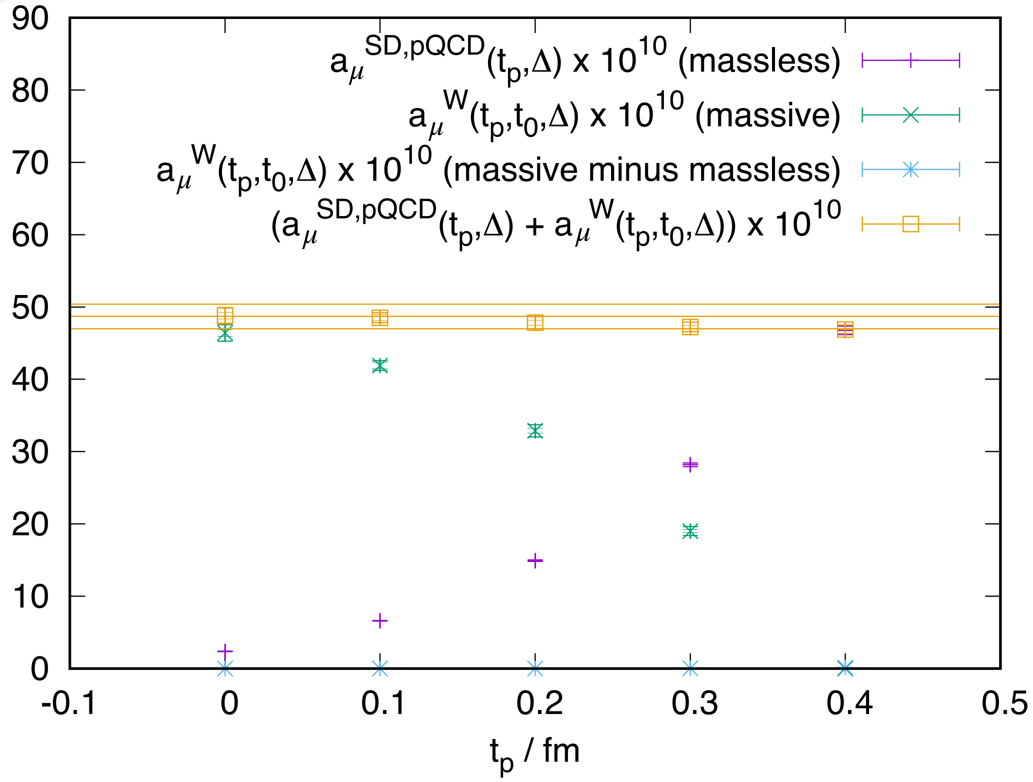


023
arXiv:2301.08696

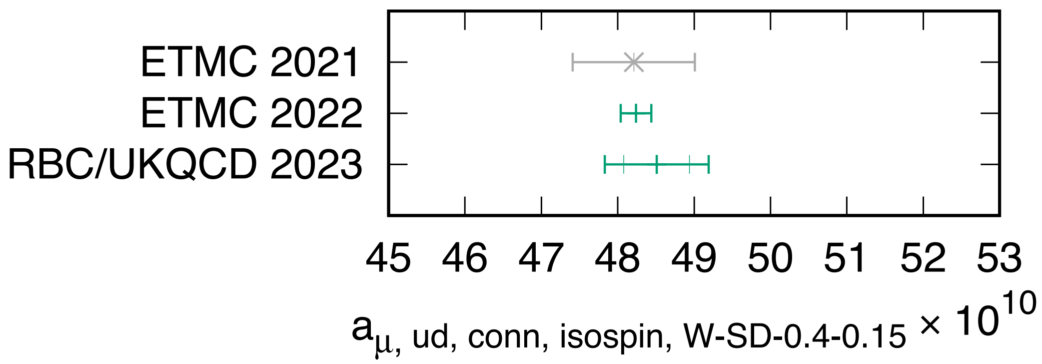
Other windows

SD

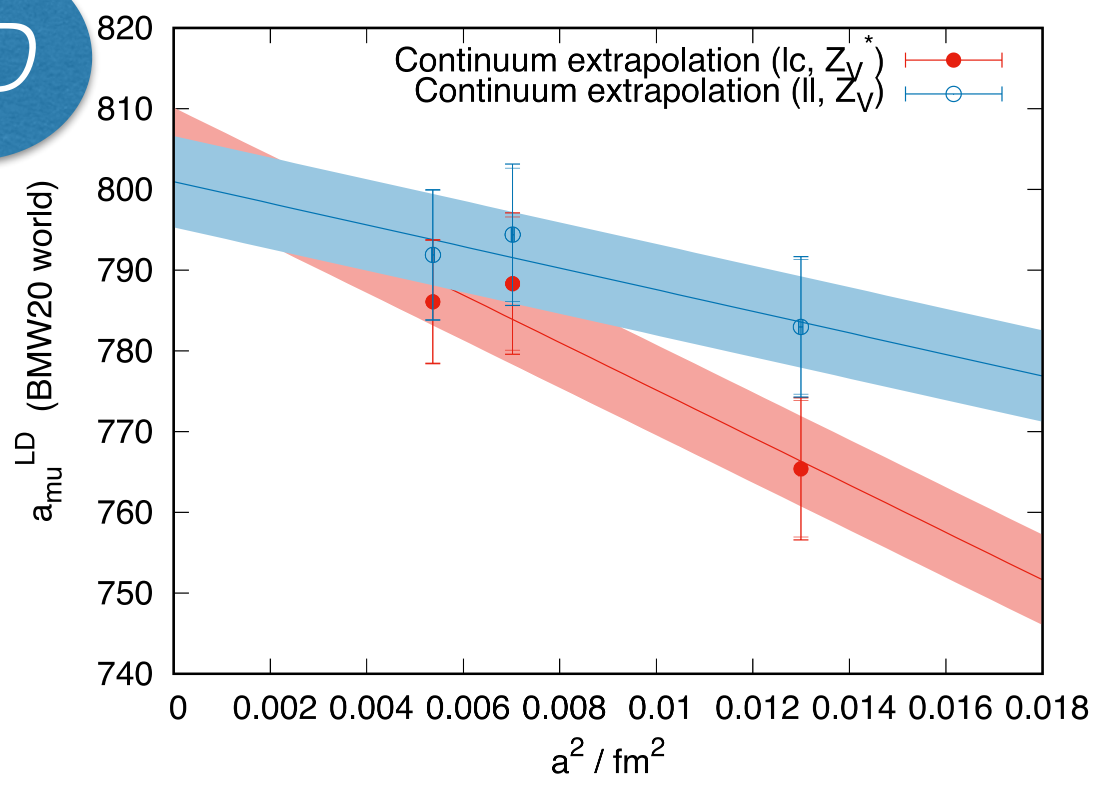
plot from RBC/UKQCD '23



- dominated by perturbation theory
- large cutoff effects
- more results expected soon



LD



RBC/UKQCD - Group A, blind, preliminary

- 5 groups, analysis in progress
- vector current is blinded allowing for a factor of 4 variation of $V(t)$
- large FV effects + StN problem
- sub-percent accuracy goal achieved

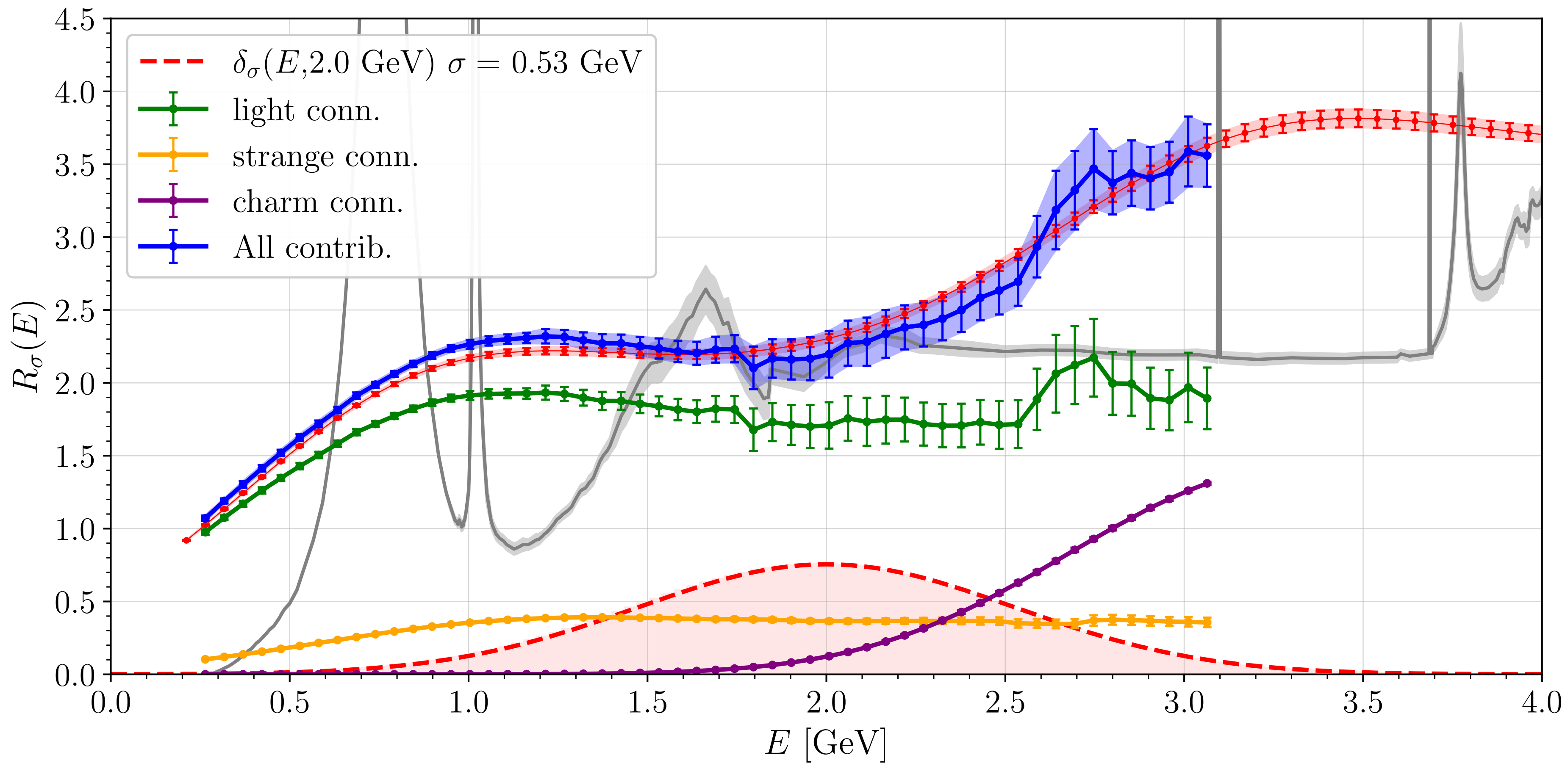
see talk by C. Lehner @ Muon g-2 TI 2023

Probing the R-ratio on the lattice

$R_\sigma(E)$: preliminary results

$$R_\sigma(E) = \int_{2M_\pi}^{\infty} d\omega \delta_\sigma(\omega, E) R(\omega) \qquad \delta_\sigma(\omega, E) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(\omega-E)^2}{2\sigma^2}}$$

$R_\sigma(E)$ from e^+e^- data



- Uncertainty coming mostly from light quark contributions, strange & charm ones are very precise
- Disconnected contributions are tiny and cannot be appreciated on this scale

Alexandrou *et al.* 2022

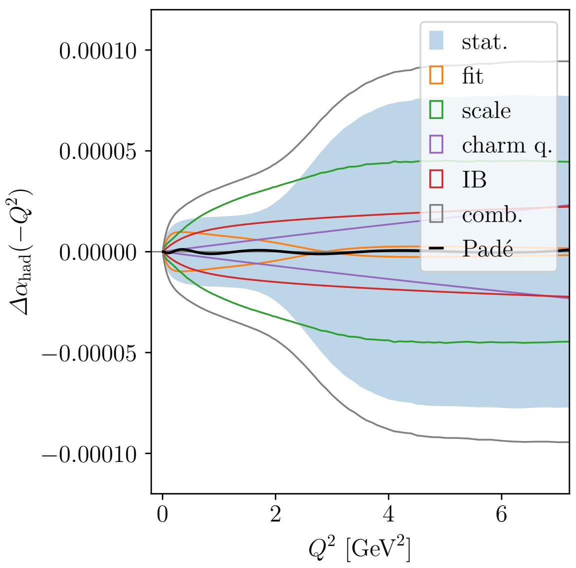
Hadronic running of α_{em} from the lattice

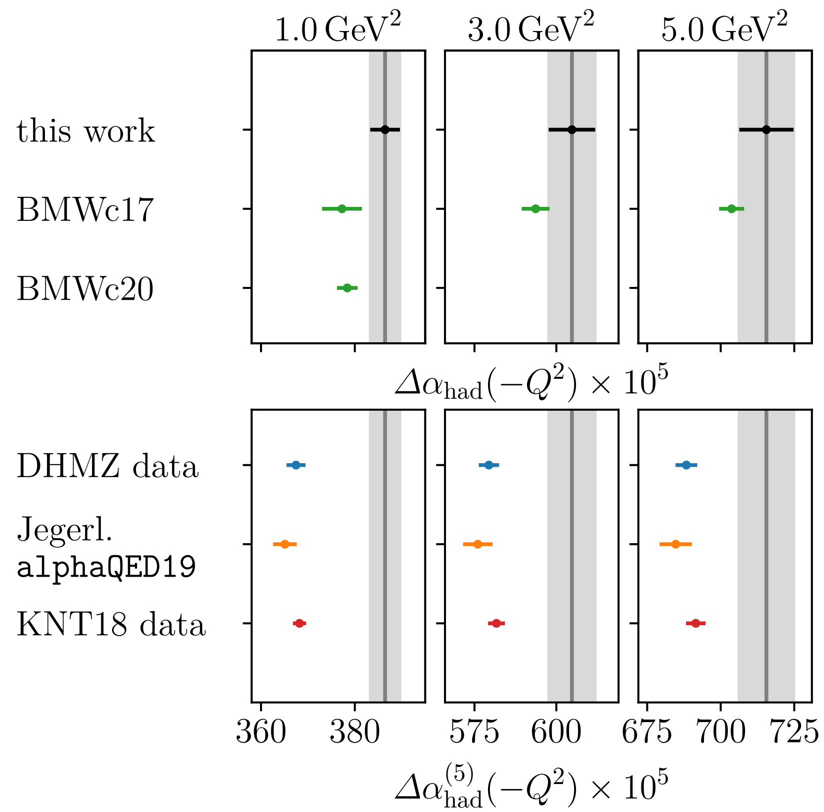
Lattice result for the hadronic running of α

[Cè et al., arXiv:2203.08676]

Starting point: Results for $\Delta\alpha_{\rm had}(-Q^2)$ for Euclidean momenta $0 \leq Q^2 \leq 7\,\mathrm{GeV}^2$ **[T. San José, TUE 17:10]**

Rational approximation:





- Mainz/CLS and BMWc (2017) differ by $2–3\%$ at the level of $1–2\sigma$
- Tension between Mainz/CLS and phenomenology by $\sim 3\sigma$ for $Q^2 \gtrsim 3\,\mathrm{GeV}^2$
- Tension increases to $\gtrsim 5\sigma$ for $Q^2 \lesssim 2\,\mathrm{GeV}^2$
  (smaller statistical error due to ansatz for continuum extrapolation)

Systematic uncertainties from fit ansatz, scale setting, charm quenching, isospin-breaking and missing bottom quark contribution (five flavour theory) included in error budget

Isospin-breaking corrections in τ-decays

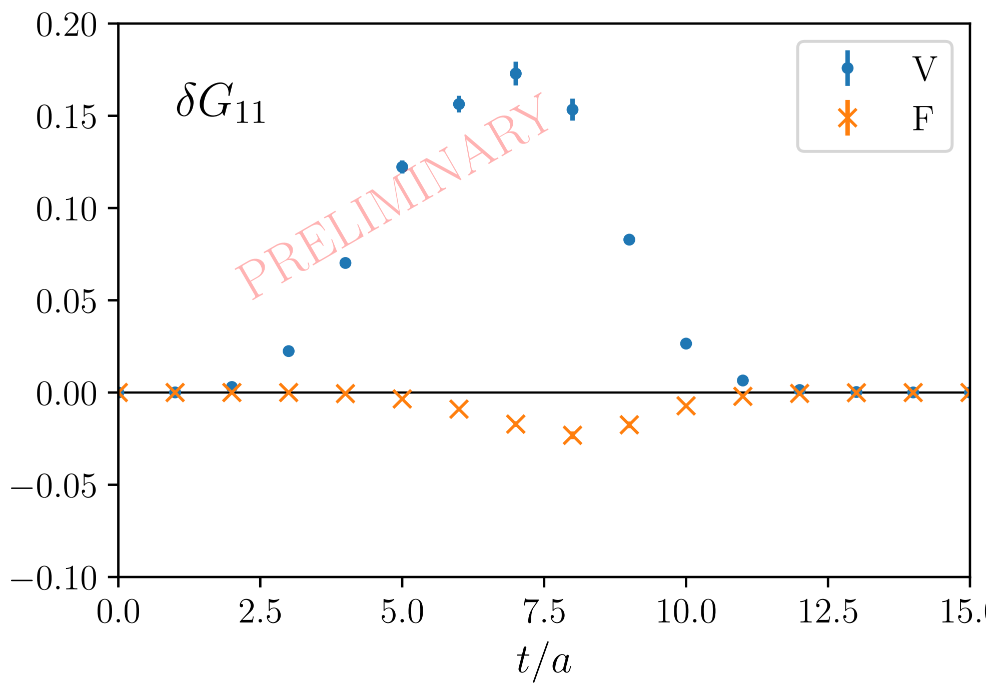
talk by M. Bruno @ Muon g-2 TI 2022

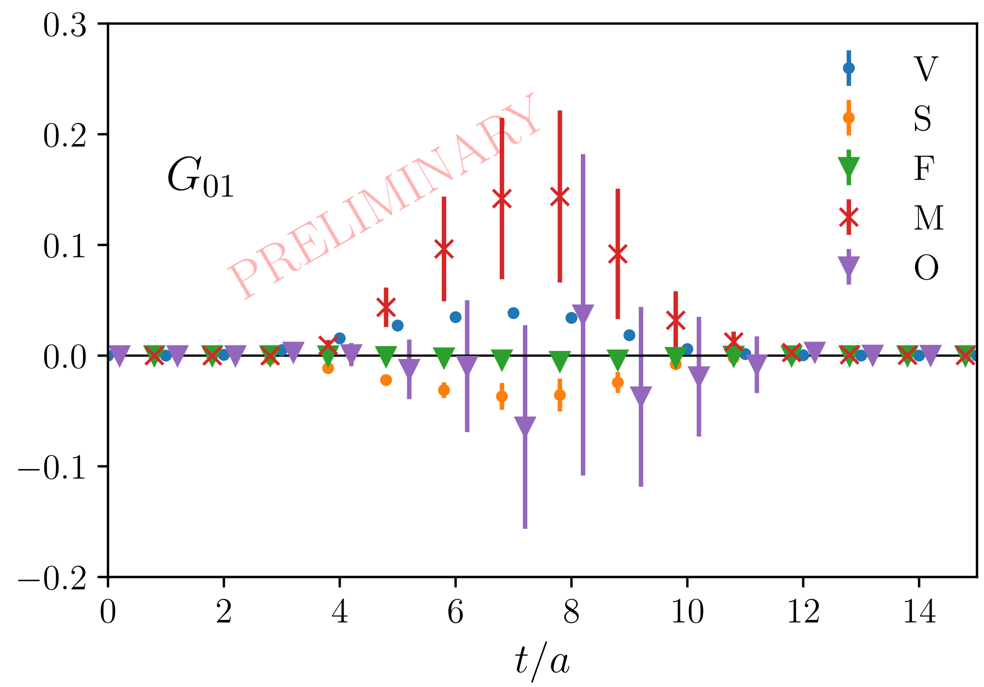
Results - Preliminary

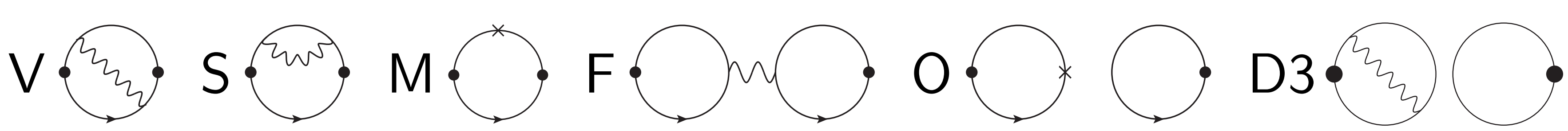
Preliminary from 48I ensemble

phys. pions, $a^{-1} \simeq 1.73$ GeV, 17 configs

cross-checks of code, data, analysis still missing

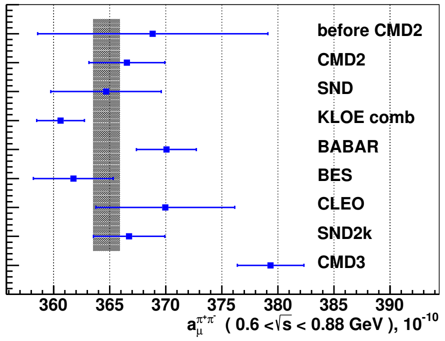






Summary and Outlook

- Tremendous progress in lattice calculations of HVP (and HLbL!) contributions
- Sub-percent calculation by BMW must be checked and impressive efforts from various lattice collaborations are in progress
- An update of the White Paper is aimed for late 2024
- Benchmark quantities (windows) crucial for checking the internal consistency of lattice calculations. For a_μ^W a new puzzle arises: remarkable agreement between lattice calculations but significant tension with dispersive prediction
- Extend calculation of window quantities to individual flavor and quark-disconnected contributions. Reach better precision for isospin-breaking contr.
- Extend comparison with phenomenological analyses to understand discrepancies. Clarify tensions in $\pi^+\pi^-$ BaBar, KLOE, CMD3



- $\mu e \rightarrow \mu e$ experiment MUonE very important for experimental cross-check and complementarity w/ LQCD