

Higher-order HVP at MUonE

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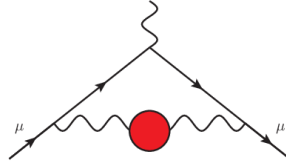
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The content is based on work done with E.Balzani and M.Passera, arXiv:2112.05704, PLB**834** 137462 (2022)

LO hadronic vacuum polarization contribution



Leading order (LO) hadronic vacuum polarization contribution to muon $g-2$.

timelike dispersive integral

spacelike dispersive integral

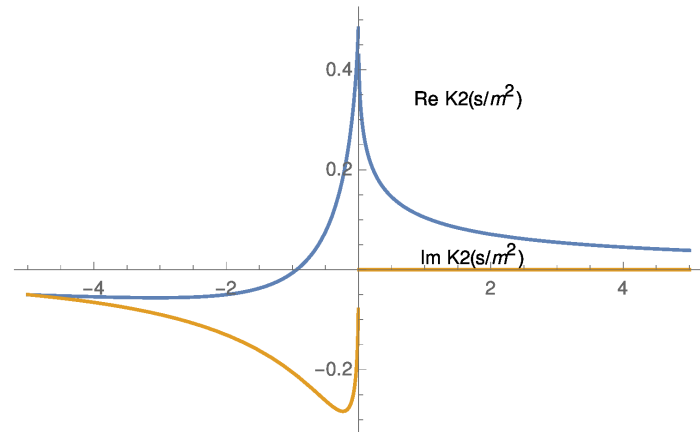
$$a_{\mu}^{\text{HVP}}(\text{LO}) = \frac{\alpha}{\pi^2} \int_{s_0=m_{\pi^0}^2}^{\infty} \frac{ds}{s} K^{(2)}(s/m^2) \text{Im}\Pi(s) = -\frac{\alpha}{\pi^2} \int_{-\infty}^0 \frac{dt}{t} \Pi(t) \text{Im}K^{(2)}(t/m^2) = 6931(40) \times 10^{-11} \text{ (WP20)}$$

$K^{(2)}(s/m^2)$: 1-loop QED $g-2$ contribution with a massive photon of mass \sqrt{s}

$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right) \ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}} \left(z - 2z^2 + \frac{z^3}{2}\right)$$

$$\text{Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z) \left[\frac{z^2}{2} - z + \frac{z-2z^2+\frac{z^3}{2}}{\sqrt{z(z-4)}} \right]$$

$$y(z) = \frac{z - \sqrt{z(z-4)}}{z + \sqrt{z(z-4)}}$$



changing variable in the dispersive integral $t \rightarrow x(y(t/m^2)) = 1 + 1/y(t/m^2)$

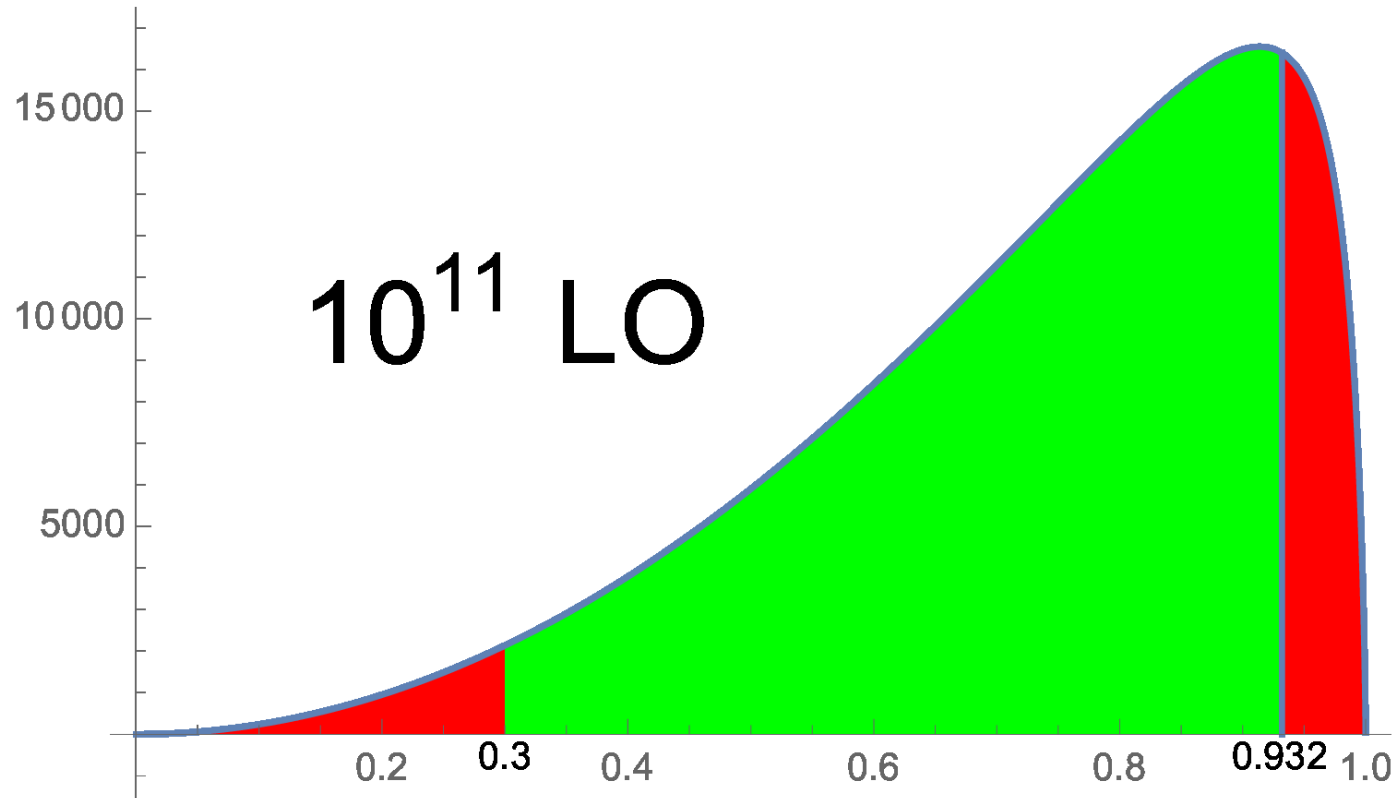
$$a_{\mu}^{\text{HVP}}(\text{LO}) = \frac{\alpha}{\pi} \int_0^1 dx \kappa^{(2)}(x) \Delta\alpha_{\text{had}}(t(x))$$

Lautrup, Peterman, deRafael 1972, Carloni Passera Trentadue Venanzoni 2015

$$\kappa^{(2)}(x) = 1 - x$$

$$\Delta\alpha_{\text{had}}(t) = -\Pi(t) \quad t(x) = m^2 \frac{x^2}{x-1}$$

LO integrand



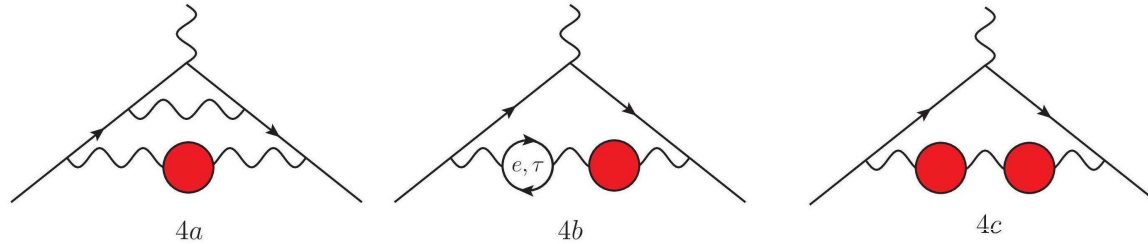
Plot of LO integrand $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\text{had}}(t(x))$ peak at $x = 0.914$

- with $E_\mu = 150\text{GeV}$ MUonE will scan the region $0.3 < x < 0.932$
- **Green**=LO scanned by MUonE= 84% of $a_\mu^{\text{HVP}}(\text{LO})$
- **Red**=LO outside MUonE

- Ignatov, Pilato, Teubner and Venanzoni, arXiv:2309.14205

It uses computation of derivatives of $\Delta\alpha_h(t)$ at $t = 0$ and suitable approximations of the time-like kernel $K_2(s)$

NLO hadronic vacuum polarization contributions



- Class a: 1 HVP insertion in one photon line of 2-loop QED vertex diagrams
- Class b: 1 HVP insertion in the photon line of 2-loop QED vertex with one electron vacuum polarization
- Class c: 2 HVP insertion in the 1-loop QED vertex diagram

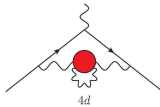
$$a_{\mu}^{\text{HVP}}(\text{NLO}; 4a) = -209.0 \times 10^{-11}$$

$$a_{\mu}^{\text{HVP}}(\text{NLO}; 4b) = +106.8 \times 10^{-11}$$

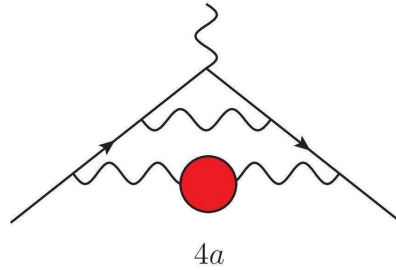
$$a_{\mu}^{\text{HVP}}(\text{NLO}; 4c) = +3.5 \times 10^{-11}$$

$$a_{\mu}^{\text{HVP}}(\text{NLO}; \text{total}) = -98.7(9) \times 10^{-11}$$

(Krause 1996, Hagiwara Liao Martin Nomura Toebner 2011, Kurz Liu Marquard Steinhauser 2014)



HVP insertion with internal corrections already incorporated in LO



timelike and spacelike integral:

$$a_{\mu}^{\text{HVP}}(\text{NLO}; 4a) = \frac{\alpha^2}{\pi^3} \int_{s_0}^{\infty} \frac{ds}{s} 2K^{(4)}(s/m^2) \text{Im}\Pi(s) = -\frac{\alpha^2}{\pi^3} \int_{-\infty}^0 \frac{dt}{t} \Pi(t) \text{Im}2K^{(4)}(t/m^2)$$

$2K^{(4)}(s/m^2)$: 2-loop QED $g-2$ contribution from diagrams with one massive photon of mass \sqrt{s} and one massless photon (factor 2 due to normalization chosen)

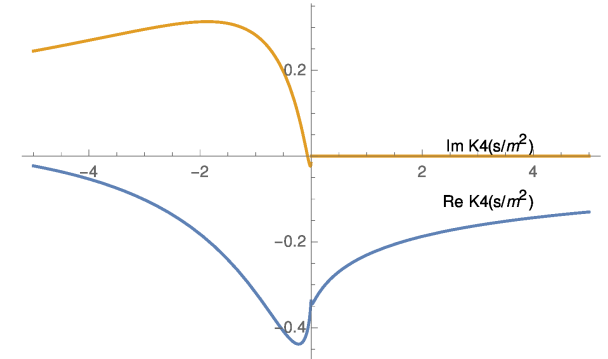
$$\text{Dispersion relations: } K^{(4)}(z) = \frac{1}{\pi} \int_{-\infty}^0 dz' \frac{\text{Im}K^{(4)}(z')}{z' - z}, \quad z > 0 \qquad \frac{1}{\pi} \int_{s_0}^{\infty} \frac{ds}{s} \frac{\text{Im}\Pi(s)}{s - q^2} = \frac{\Pi(q^2)}{q^2}, \quad q^2 < 0$$

$$\begin{aligned}
 K^{(4)}(z) = & \left(\frac{z^2}{2} - \frac{7z}{6} + \frac{1}{2} \right) \left[-3\text{Li}_3(-y) - 6\text{Li}_3(y) + 2(\text{Li}_2(-y) + 2\text{Li}_2(y)) \ln y + \frac{1}{2}(\ln^2 y + \pi^2) \ln(y+1) + \ln(1-y) \ln^2 y \right] \\
 & + \frac{\left(-\frac{z^3}{6} + \frac{z^2}{4} - \frac{7z}{6} - \frac{4}{z-4} + \frac{13}{3} \right) \left(\text{Li}_2(-y) + \frac{\ln^2 y + \pi^2}{4} \right)}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{12} + \frac{17z^2}{6} - 2z \right) \left(\text{Li}_2(y) - \frac{1}{4} \ln^2 y + \ln(1-y) \ln y - \frac{\pi^2}{6} \right)}{\sqrt{(z-4)z}} \\
 & + \left(-\frac{29z^2}{96} + \frac{53z}{48} + \frac{2}{z-4} - \frac{1}{3z} + \frac{19}{24} \right) \ln^2 y + \frac{\left(\frac{23z^3}{144} - \frac{115z^2}{72} + \frac{127z}{36} - \frac{4}{3} \right) \ln y}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{48} + \frac{17z^2}{24} - \frac{z}{2} \right) \ln y \ln z}{\sqrt{(z-4)z}} \\
 & + \frac{1}{6} \pi^2 \left(-\frac{z^2}{2} + \frac{5z}{24} - \frac{2}{z} + \frac{9}{4} \right) + \frac{5}{96} z^2 \ln^2 z + \left(\frac{23z^2}{144} - \frac{7z}{36} + \frac{1}{z-4} + \frac{19}{12} \right) \ln z + \frac{115z}{72} - \frac{139}{144} \quad \text{Barbieri Remiddi 1975}
 \end{aligned}$$

$$K^{(4)}(0) = \frac{197}{144} + \frac{1}{12} \pi^2 - \frac{1}{2} \pi^2 \ln 2 + \frac{3}{4} \zeta(3) = -0.328479 \quad \text{2-loop } g-2 \quad K^{(4)}(z \gg 1) \rightarrow \frac{1}{z} \left(-\frac{23 \ln(z)}{36} - \frac{\pi^2}{3} + \frac{223}{54} \right)$$

$$\text{Im}K^{(4)}(z + i\epsilon) = \pi \theta(-z) F^{(4)}(1/y(z)) \quad y(z) = \frac{z - \sqrt{z(z-4)}}{z + \sqrt{z(z-4)}} < -1$$

$$\begin{aligned}
 F^{(4)}(u) = & \frac{-3u^4 - 5u^3 - 7u^2 - 5u - 3}{6u^2} \left(2\text{Li}_2(-u) + 4\text{Li}_2(u) + \ln(-u) \ln((1-u)^2(u+1)) \right) \\
 & + \frac{(u+1)(-u^3 + 7u^2 + 8u + 6)}{12u^2} \ln(u+1) + \frac{(-7u^4 - 8u^3 + 8u + 7)}{12u^2} \ln(1-u) \\
 & + \frac{23u^6 - 37u^5 + 124u^4 - 86u^3 - 57u^2 + 99u + 78}{72(u-1)^2 u(u+1)} \\
 & + \frac{12u^8 - 11u^7 - 78u^6 + 21u^5 + 4u^4 - 15u^3 + 13u + 6}{12(u-1)^3 u(u+1)^2} \ln(-u)
 \end{aligned}$$



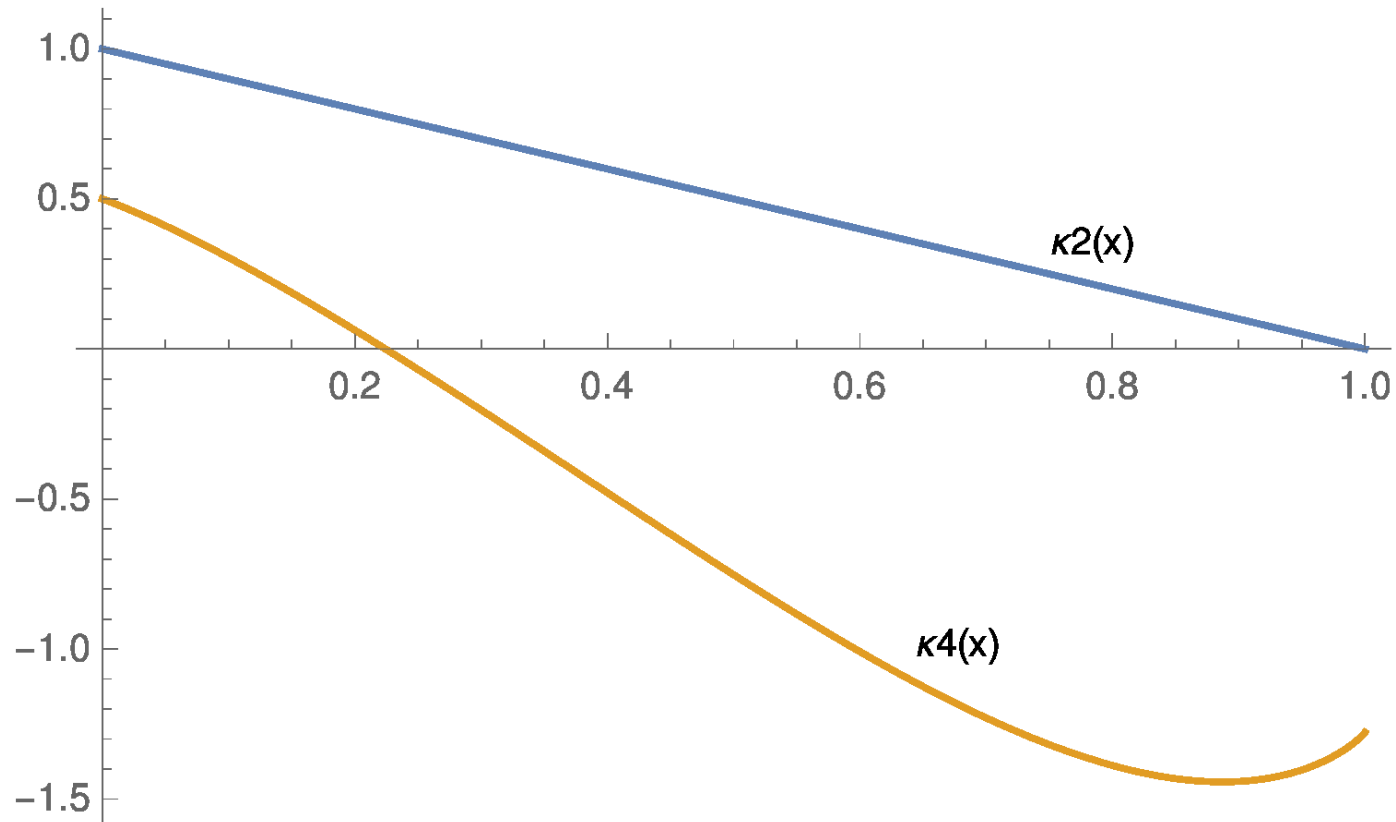
Balzani, S.L., Passera 2112.05704, Nesterenko 2112.05009.

$$a_\mu^{\text{HVP}}(\text{NLO}; 4a) = \left(\frac{\alpha}{\pi} \right)^2 \int_0^1 dx \kappa^{(4)}(x) \Delta\alpha_{\text{had}}(t(x))$$

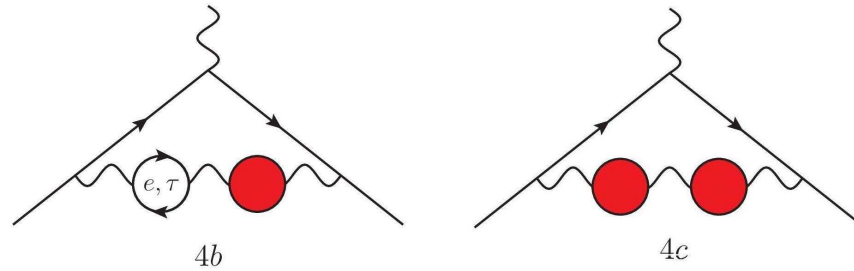
$z \rightarrow y \rightarrow x$ Space-like NLO kernel $\kappa^{(4)}(x)$

$$\kappa^{(4)}(x) = \frac{2(2-x)}{x(x-1)} F^{(4)}(x-1)$$

Comparison of the LO and NLO class $4a$ x -kernels



- $\kappa^{(4)}(1) = -\frac{23}{18}$, $\kappa^{(4)}(0) = \frac{1}{2}$;
- $\kappa^{(4)}(x)$ provides stronger weight a large $q^2 < 0$ ($x \rightarrow 1$) than $\kappa^{(2)}(x)$



starting from time-like expressions one finds

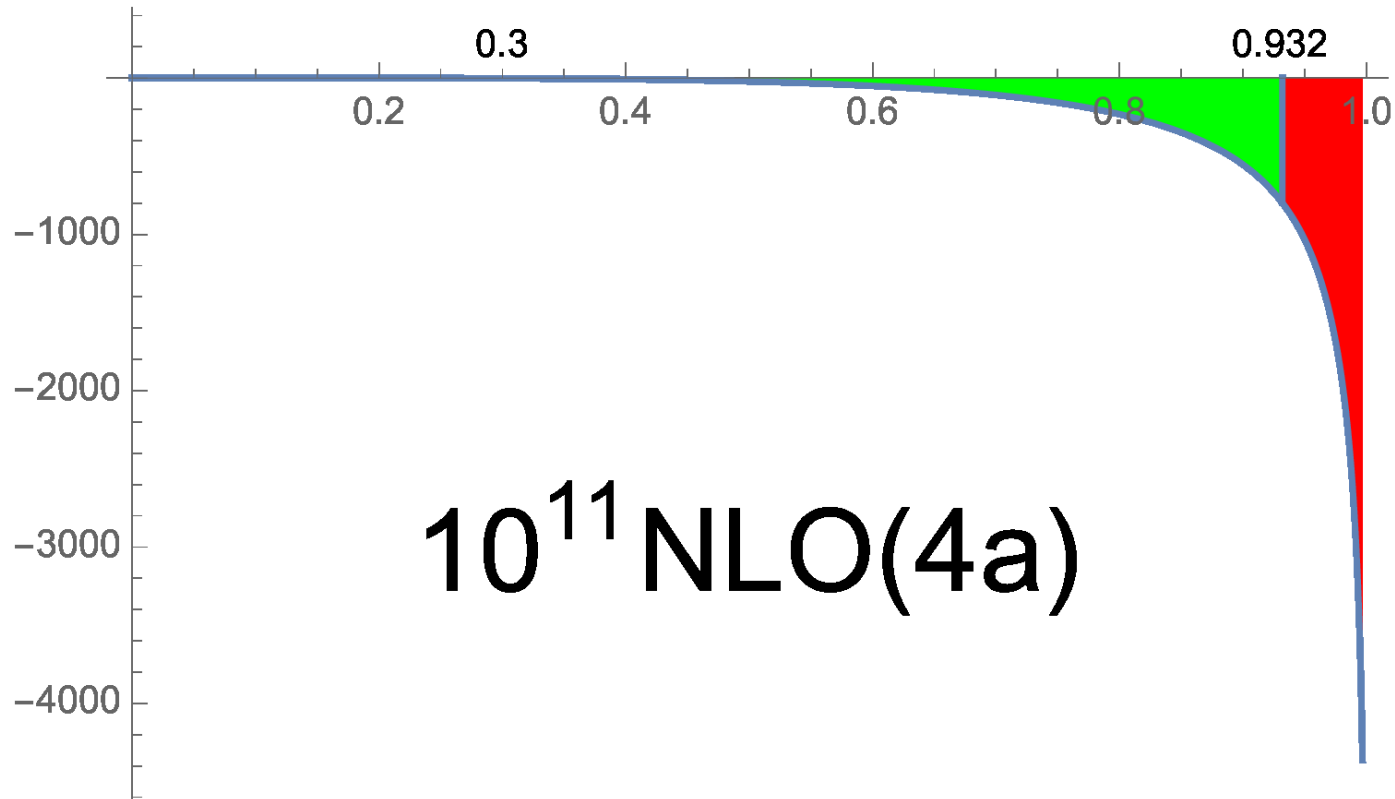
$$a_{\mu}^{\text{HVP}}(\text{NLO}; 4b) = \frac{\alpha}{\pi} \int_0^1 dx \kappa^{(2)}(x) \Delta\alpha_{\text{had}}(t(x)) 2 \left(\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_{\tau}^{(2)}(t(x)) \right)$$

$$a_{\mu}^{\text{HVP}}(\text{NLO}; 4c) = \frac{\alpha}{\pi} \int_0^1 dx \kappa^{(2)}(x) (\Delta\alpha_{\text{had}}(t(x)))^2$$

$\Pi_l^{(2)}(t) = -\Delta\alpha_l(t)$ renormalized one-loop QED vacuum polarization function

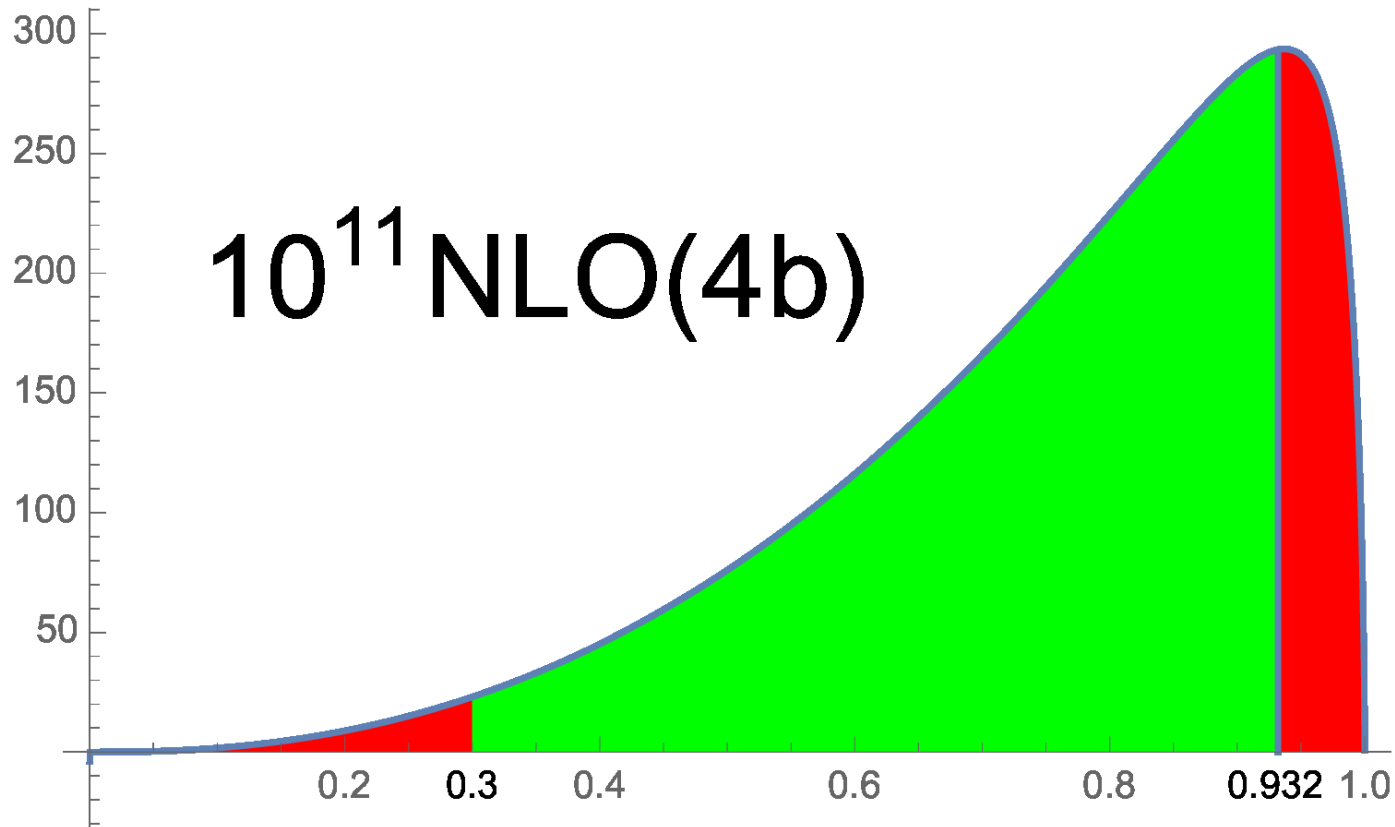
$$\Pi_l^{(2)}(t) = \left(\frac{\alpha}{\pi} \right) \left[\frac{8}{9} - \frac{\beta_l^2}{3} + \beta_l \left(\frac{1}{2} - \frac{\beta_l^2}{6} \right) \ln \frac{\beta_l - 1}{\beta_l + 1} \right], \quad \beta_l = \sqrt{1 - 4m_l^2/t}$$

NLO integrand class 4a



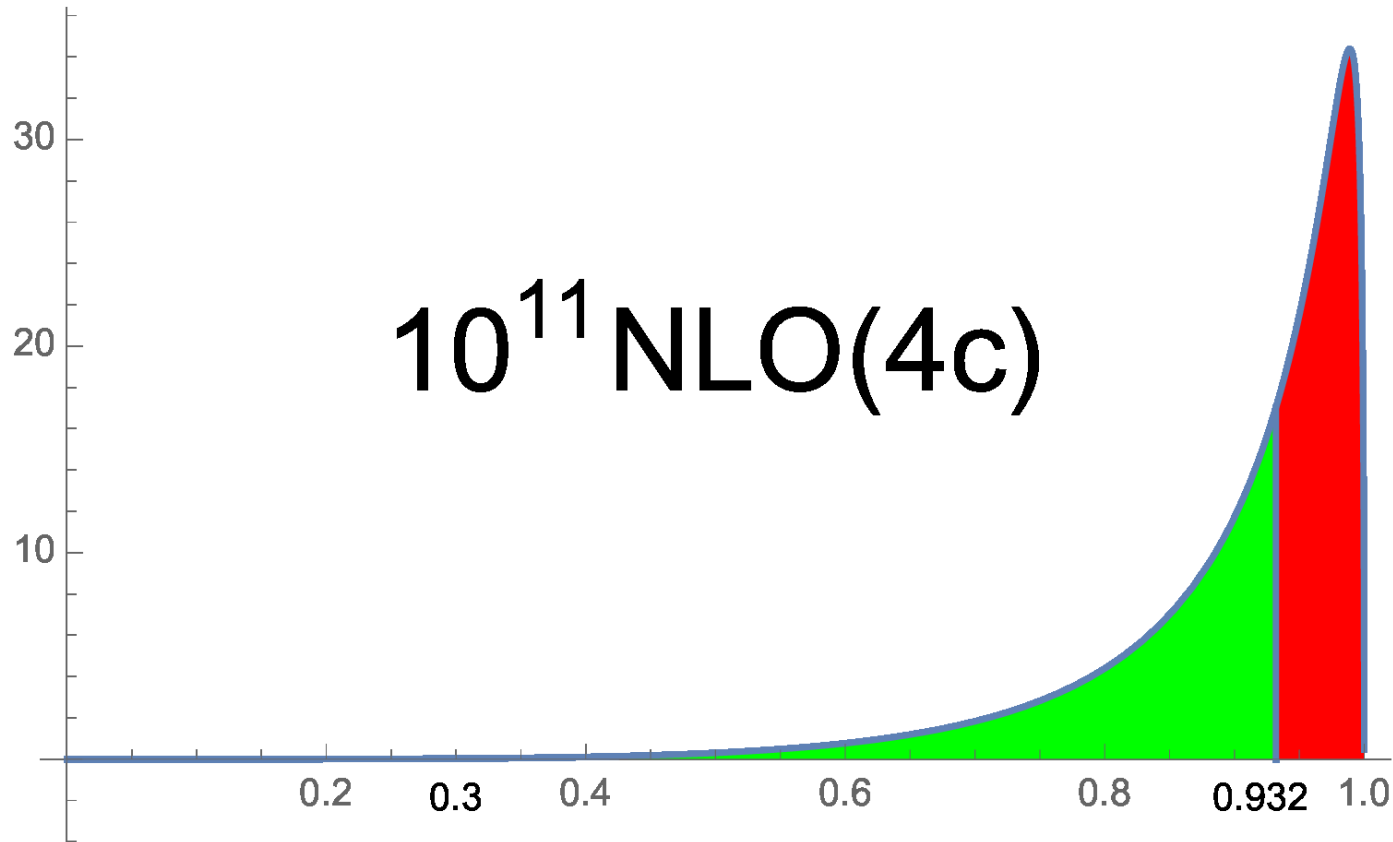
Plot of NLO(4a) integrand $(\alpha/\pi)^2 \kappa^{(4)}(x) \Delta\alpha_{\text{had}}(t(x))$ logarithmic singularity for $x = 1$

- with $E_\mu = 150\text{GeV}$ MUonE will scan the region $0.3 < x < 0.932$
- **Green**=NLO(4a) contribution scanned by MUonE= 41% of $a_\mu^{\text{HVP}}(\text{NLO}; 4a)$
- **Red**=NLO(4a) outside MUonE



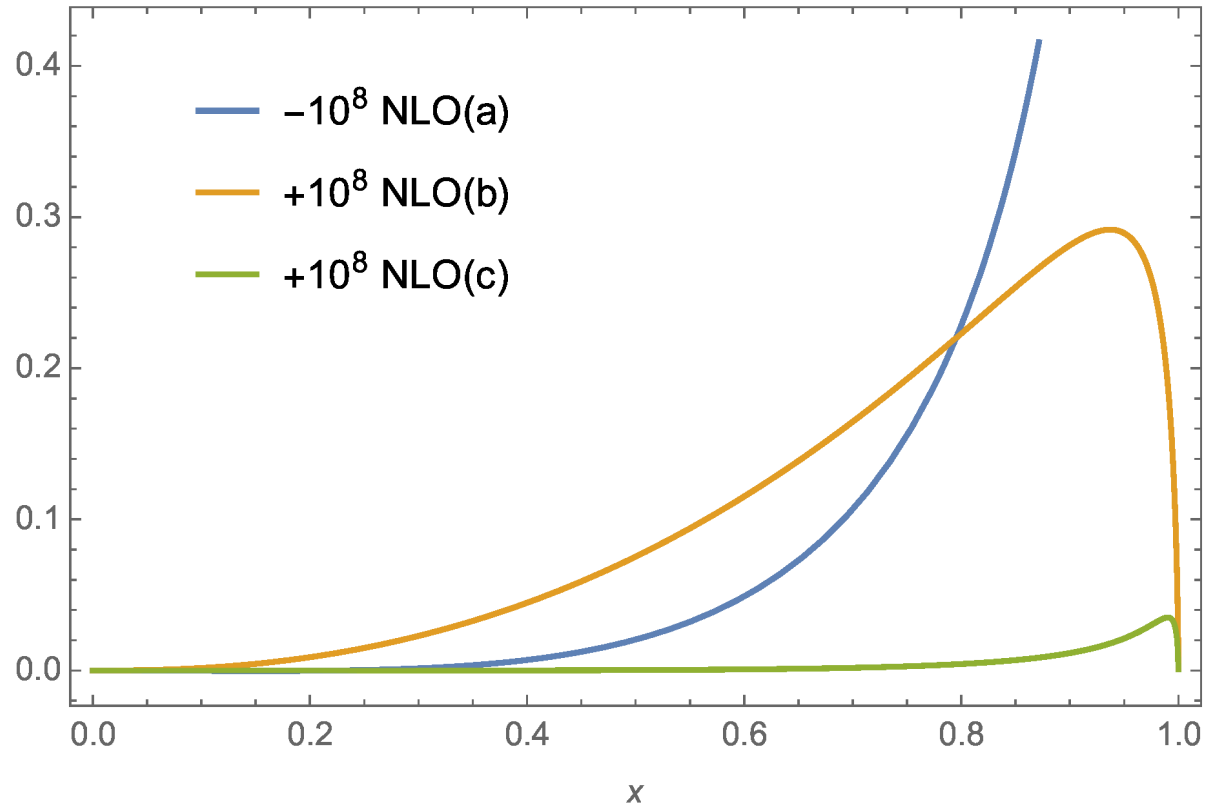
Plot of NLO(4b) integrand $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\text{had}}(t(x)) 2 \left(\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x)) \right)$ peak at $x = 0.936$

- with $E_\mu = 150\text{GeV}$ MUonE will scan the region $0.3 < x < 0.932$
- **Green**=NLO(4b) contribution scanned by MUonE= 82% of $a_\mu^{\text{HVP}}(\text{NLO}; 4b)$
- **Red**=NLO(4b) outside MUonE



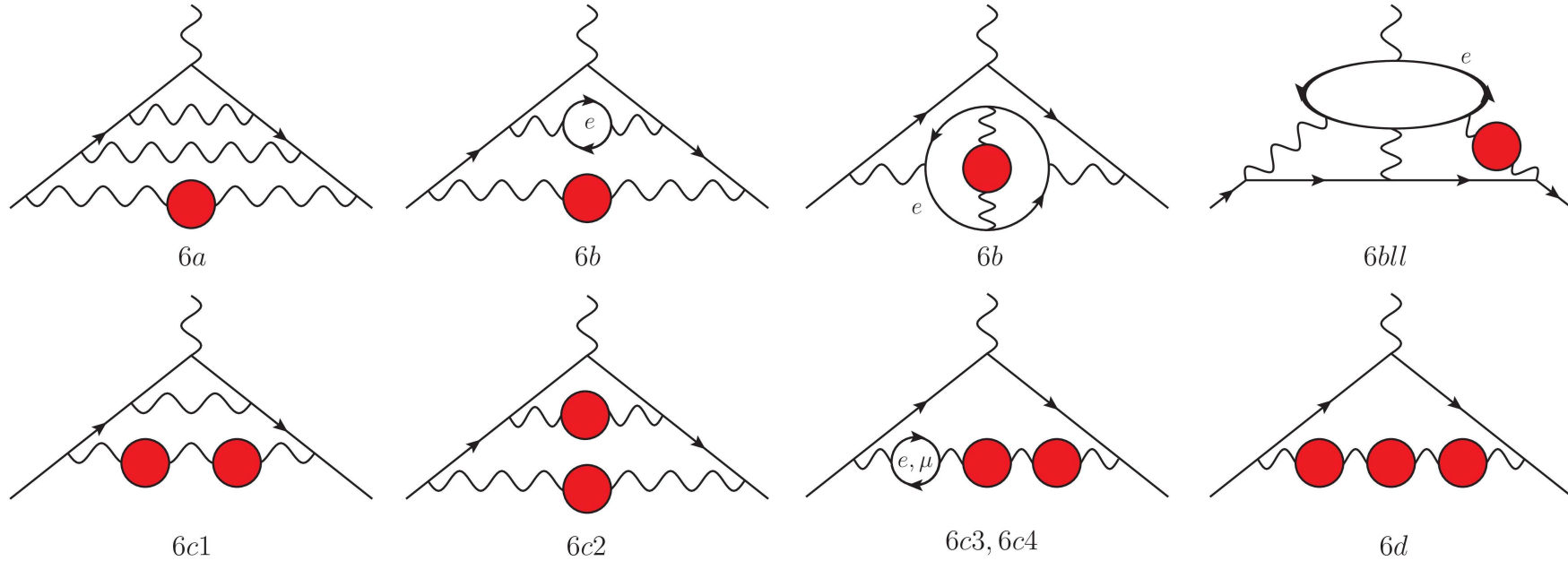
Plot of NLO(4c) integrand $(\alpha/\pi)\kappa^{(2)}(x) (\Delta\alpha_{\text{had}}(t(x)))^2$ peak at $x = 0.988$

- with $E_\mu = 150\text{GeV}$ MUonE will scan the region $0.3 < x < 0.932$
- **Green**=NLO(4c) contribution scanned by MUonE= 49% of $a_\mu^{\text{HVP}}(\text{NLO}; 4c)$
- **Red**=NLO(4c) outside MUonE



the 3 NLO integrands

NNLO hadronic vacuum polarization contributions



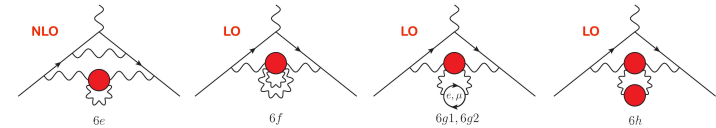
sample NNLO diagrams

- set 6a contains also diagrams with muon loops (like 6b 6bll)

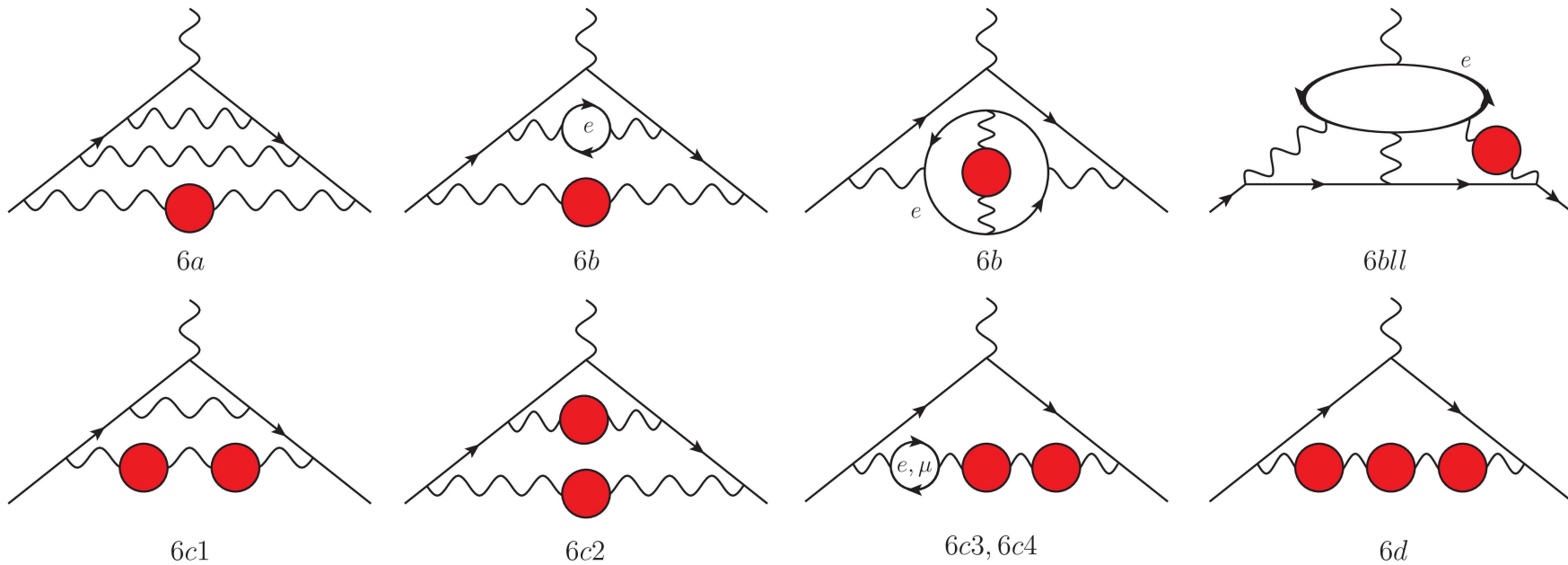
- HVP with internal corrections already incorporated in NLO and LO

- $a_{\mu}^{\text{HVP}}(\text{NNLO}; \text{total}) = +12.4(1) \times 10^{-11}$

Kurz Liu Marquard Steinhauser 2014



NNLO hadronic vacuum polarization contributions



$$a_{\mu}^{\text{HVP}}(\text{NNLO}; 6a) = \frac{\alpha^3}{\pi^4} \int_{m_{\mu}^2}^{\infty} \frac{ds}{s} K^{(6)}(s/m^2) \text{Im}\Pi(s) = -\frac{\alpha^3}{\pi^4} \int_{-\infty}^0 \frac{dt}{t} \Pi(t) \text{Im}K^{(6)}(t/m^2)$$

- At NLO, $K^{(4)}(s/m^2)$ and $\text{Im}K^{(4)}(s/m^2)$ are known exactly.
- Unfortunately at NNLO $K^{(6)}(s/m^2)$ is **NOT** known analytically.
- Only a few terms of the asymptotic expansions for large s are known.
- We need to find *approximated* spacelike kernels from the asymptotic expansions

$K^{(6a)}(s/m^2)$: Only the **first 4 terms** of the expansion in power series of $r = m^2/s$ are **known** $\rightarrow n=4$

Kurz, Liu, Marquard, Steinhauser, PLB734 (2014) 144

The expansion in small r contain terms with $r^n \ln r$, $r^n \ln^2 r$ and $r^n \ln^3 r$. As in NLO, we use an integral ansatz:

$$K^{(6a)}(s/m^2) = r \int_0^1 d\xi \left[\frac{L^{(6a)}(\xi)}{\xi + r} + \frac{P^{(6a)}(\xi)}{1 + r\xi} \right] \quad L^{(6a)}(\xi) = G^{(6a)}(\xi) + H^{(6a)}(\xi) \ln \xi + J^{(6a)}(\xi) \ln^2 \xi \quad \text{new@NNLO}$$

$G^{(6a)}$, $H^{(6a)}$, $J^{(6a)}$, $P^{(6a)}$ polynomials of degree 3

$$G^{(6a)}(\xi) = \sum_{i=0}^3 g_i^{(6a)} \xi^i, \quad H^{(6a)}(\xi) = \sum_{i=0}^3 h_i^{(6a)} \xi^i, \quad J^{(6a)}(\xi) = \sum_{i=0}^3 j_i^{(6a)} \xi^i, \quad P^{(6a)}(\xi) = \sum_{i=0}^3 p_i^{(6a)} \xi^i$$

We integrate in ξ , expand in r , and we **fit the coefficients** $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$ and $p_i^{(6a)}$, $i = 0, 1, 2, 3$, in order to match the coefficients of the asymptotic expansion in r of $K^{(6a)}(s/m^2)$. Then approximated kernel $\bar{\kappa}^{(6a)}(x)$ is

$$a_\mu^{HVP}(\text{NNLO}; 6a) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 dx \bar{\kappa}^{(6a)}(x) \Delta\alpha_{\text{had}}(t(x)),$$

$$\bar{\kappa}^{(6a)}(x) = \begin{cases} \frac{2-x}{x(1-x)} P^{(6a)}\left(\frac{x^2}{1-x}\right), & 0 < x < x_\mu = (\sqrt{5} - 1)/2 = 0.618\dots \\ \frac{2-x}{x^3} L^{(6a)}\left(\frac{1-x}{x^2}\right), & x_\mu < x < 1 \quad \text{discontinuous in } x_\mu \end{cases}$$

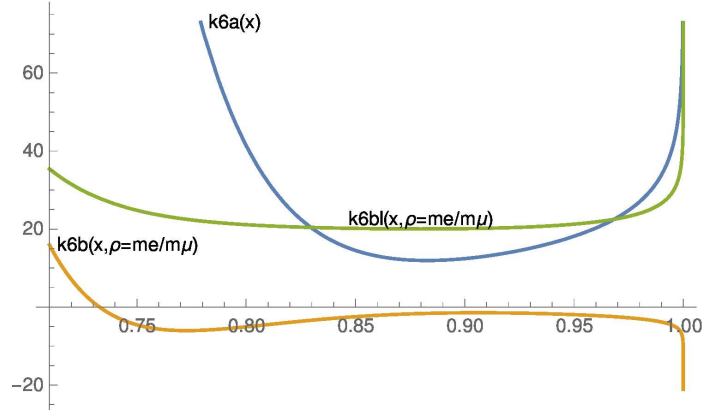
- The contributions of classes (6b) and (6bll) can be calculated similarly to class (6a).
- $a_\mu^{HVP}(\text{NNLO}; 6a) = +8.0 \times 10^{-11}$ $a_\mu^{HVP}(\text{NNLO}; 6b) = -4.1 \times 10^{-11}$ $a_\mu^{HVP}(\text{NNLO}; 6bll) = +9.1 \times 10^{-11}$
- The uncertainty due to the series approximations of $K^{(6a)}$, $K^{(6b)}$, $K^{(6bll)}$ are estimated to be less than $O(10^{-12})$

(6a)	
$j_0 = 0;$	$h_0 = -\frac{359}{36};$
$j_1 = -\frac{3793}{864};$	$h_1 = \frac{122293}{5184};$
$j_2 = \frac{35087}{21600};$	$h_2 = -\frac{43879427}{648000};$
$j_3 = \frac{1592093}{43200};$	$h_3 = \frac{14388407}{48000};$
$g_0 = \frac{1301}{144} - \frac{19\pi^2}{9};$	
$g_1 = \frac{441277}{10368} + \pi^2 \left(-\frac{355}{648} + \ln 4 \right) + \frac{25}{2} \zeta(3);$	
$g_2 = -\frac{5051645167}{38880000} + \pi^2 \left(\frac{221411}{32400} - 18 \ln 2 \right) - \frac{3919}{60} \zeta(3);$	
$g_3 = \frac{14588342017}{38880000} + \pi^2 \left(-\frac{2479681}{64800} + 112 \ln 2 \right) + \frac{3113}{10} \zeta(3);$	
$p_0 = -\frac{1808080780513}{14580000} + \frac{41851\pi^4}{15} + \frac{8432\ln^4 2}{3} + 67456 a_4 + \frac{2085448}{15} \zeta(3) + \pi^2 \left(-\frac{11944163099}{194400} + \frac{272}{3} (180 - 31 \ln 2) \ln 2 + \frac{115072}{3} \zeta(3) \right) - \frac{575360}{3} \zeta(5);$	
$p_1 = \frac{134017456919}{96000} - \frac{4481182\pi^4}{135} - \frac{98420\ln^4 2}{3} - 787360 a_4 + 2255200 \zeta(5) + \pi^2 \left(\frac{23549054249}{32400} - 201122 \ln 2 + \frac{98420\ln^2 2}{3} - 451040 \zeta(3) \right) - \frac{57189259}{36} \zeta(3);$	
$p_2 = -\frac{13069081405453}{3888000} + \frac{330073\pi^4}{4} + 80790 \ln^4 2 + 1938960 a_4 + \frac{77371609}{20} \zeta(3) + \pi^2 \left(-\frac{729995599}{405} + 6(85313 - 13465 \ln 2) \ln 2 + 1114360 \zeta(3) \right) - 5571800 \zeta(5);$	
$p_3 = \frac{1274611832039}{583200} - \frac{986377\pi^4}{15} - 53340 \ln^4 2 - 1280160 a_4 + \frac{11057200}{3} \zeta(5) + \pi^2 \left(\frac{5809559289}{4860} + 420 \ln 2 (-823 + 127 \ln 2) - \frac{2211440}{3} \zeta(3) \right) - \frac{22833188}{9} \zeta(3);$	

Table 1: The coefficients $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$, $p_i^{(6a)}$ ($i = 0, 1, 2, 3$). The superscript (6a) has been dropped for simplicity. In the above coefficients, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$ and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \text{Li}_4(1/2)$.

(6b)	
$j_0 = 0;$	$h_0 = \frac{65}{54};$
$j_1 = \frac{11}{27};$	$h_1 = -\frac{3559}{1296} + \rho^2 + \frac{5}{18} \ln \rho;$
$j_2 = \frac{41}{120};$	$h_2 = \frac{3917}{432} - \frac{82\rho^2}{3} + \frac{61}{10} \ln \rho;$
$j_3 = -\frac{507}{40};$	$h_3 = -\frac{4109}{80} + \frac{2211\rho^2}{10} - \frac{1763}{30} \ln \rho;$
$g_0 = \frac{1}{108} (259 - 72\rho^2 + 276 \ln \rho);$	
$g_1 = -\frac{9215}{1296} + \frac{65\pi^2}{162} - \frac{3\rho^2}{4} + \frac{49\rho^2}{36} + \left(-\frac{301}{54} + 8\rho^2 \right) \ln \rho + \frac{4}{3} \ln^2 \rho + 2 \zeta(3);$	
$g_2 = \frac{501971}{40500} - \frac{113\pi^2}{36} + \frac{270\rho^2}{36} - \frac{8417\rho^2}{180} + \left(\frac{3479}{900} - 44\rho^2 \right) \ln \rho - 8 \ln^2 \rho - 12 \zeta(3);$	
$g_3 = -\frac{2523823}{324000} + \frac{625\pi^2}{36} - 49\rho^2 + \frac{84946\rho^2}{225} + \left(\frac{987}{50} + 200\rho^2 \right) \ln \rho + \frac{112}{3} \ln^2 \rho + 56 \zeta(3);$	
$p_0 = -\frac{95519053063}{486000} - 7275\pi^2 \rho + \left(-\frac{587150693}{5400} + \frac{75272\rho^2}{3} + \frac{120800\pi^2}{9} \right) \ln \rho + \left(\frac{1135508}{9} + 96\rho^2 \right) \zeta(3) + 4720 \ln^2 \rho + \frac{1067115409\rho^2}{5400} + \pi^2 \left(\frac{24382331}{810} - \frac{285184}{309} \ln 2 \right) - 32\pi^2 \rho^2 (687 + \ln 4);$	
$p_1 = \frac{279489782879}{121500} + \frac{179283\pi^2 \rho}{2} + \left(\frac{2280933773}{1800} - 309540\rho^2 - 1419328\pi^2 \right) \ln \rho - \frac{10}{3} (446023 + 216\rho^2) \zeta(3) + \frac{174712}{3} \ln^2 \rho - \frac{174350167\rho^2}{75} + \pi^2 \left(-\frac{143574463}{405} + \frac{3352256 \ln 2}{9} \right) + \frac{16}{3} \pi^2 \rho^2 (48481 + 90 \ln 2);$	
$p_2 = -\frac{229560199193}{40500} - \frac{912495\pi^2 \rho}{4} + \left(-\frac{1867939691}{600} + 788488\rho^2 + \frac{1168336\pi^2}{3} \right) \ln \rho + \left(\frac{11034553}{3} + 1440\rho^2 \right) \zeta(3) + 148348 \ln^2 \rho + \frac{258653648\rho^2}{45} + \frac{4}{135} \pi^2 (29597029 - 31048560 \ln 2) - \frac{320}{3} \pi^2 \rho^2 (5989 + \ln 512);$	
$p_3 = \frac{72762177677}{19440} + 154035\pi^2 \rho - \frac{7}{108} (-31650719 + 3973440\pi^2 + 8220240\rho^2) \ln \rho - \frac{280}{9} (78283 + 27\rho^2) \zeta(3) + \frac{100240}{3} \ln^2 \rho - \frac{513692207\rho^2}{135} + \frac{35}{162} \pi^2 (-2687659 + 2816064 \ln 2) + \frac{140}{3} \pi^2 \rho^2 (9055 + \ln 4096);$	

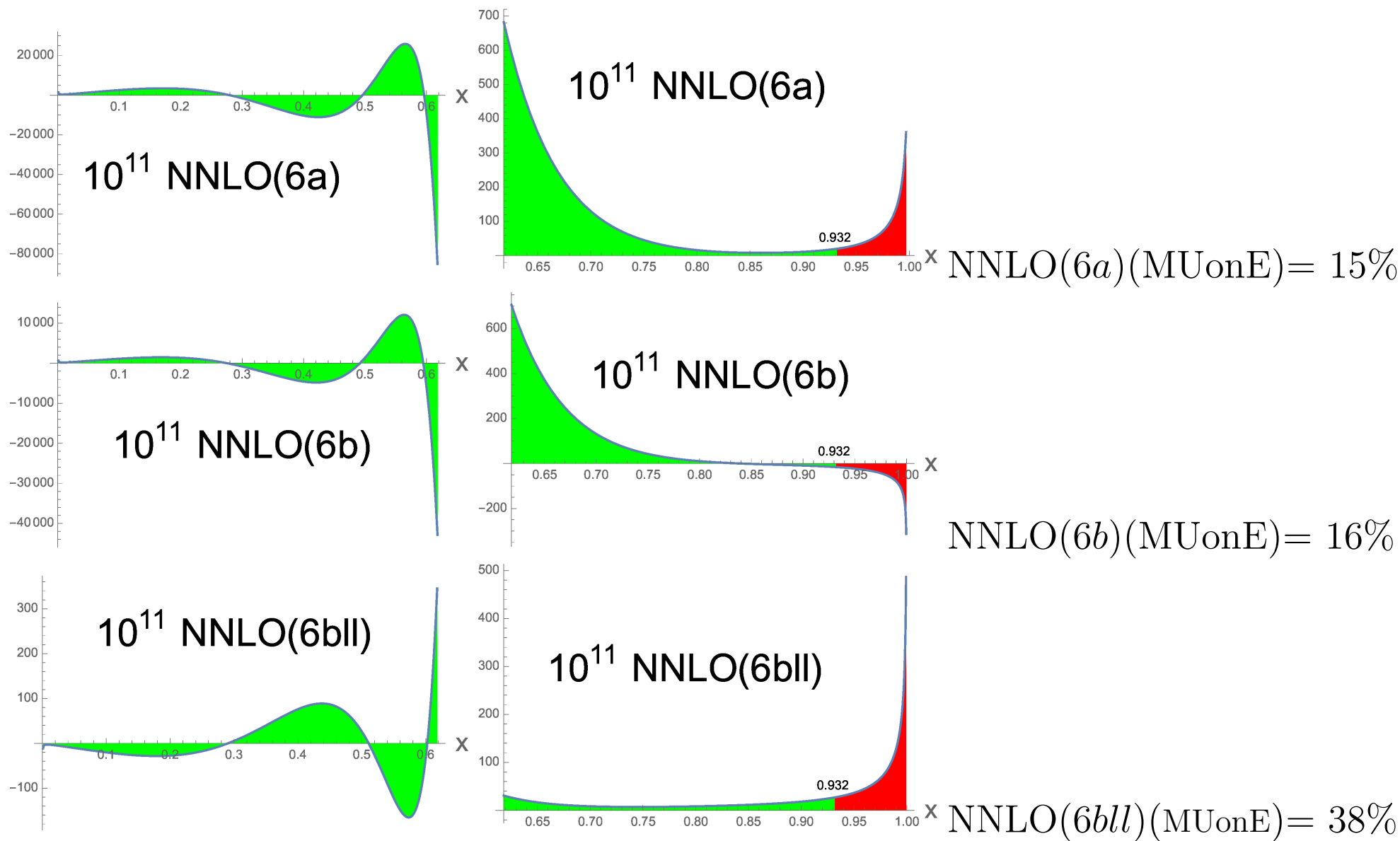
Table 2: The coefficients $g_i^{(6b)}$, $h_i^{(6b)}$, $j_i^{(6b)}$, $p_i^{(6b)}$ ($i = 0, 1, 2, 3$). The superscript (6b) has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \text{Li}_4(1/2)$.



(6bl)	
$j_0 = 0;$	$h_0 = -\frac{9}{2};$
$j_1 = \frac{4}{27} - \frac{9\rho^2}{2};$	$h_1 = \frac{59}{9} - \frac{275\rho^2}{36} - 18\rho^2 \ln \rho;$
$j_2 = -\frac{41}{48} + \frac{2201\rho^2}{216};$	$h_2 = -\frac{485}{32} + \frac{1351\rho^2}{48} + \frac{659\rho^2}{18} \ln \rho;$
$j_3 = \frac{3037}{900} - \frac{5909\rho^2}{216};$	$h_3 = \frac{282617}{6750} - \frac{10481\rho^2}{108} - \frac{851\rho^2}{9} \ln \rho;$
$g_0 = \frac{43}{8} - 4\pi^2 \rho + 15\rho^2 + \pi^2 \rho^2 - 18\rho^2 \ln \rho + 6\rho^2 \ln^2 \rho;$	
$g_1 = -\frac{73}{81} + \frac{8\pi^2}{81} + \frac{40\pi^2 \rho}{9} + \frac{2437\rho^2}{108} + \frac{17\pi^2 \rho^2}{9} + \frac{607\rho^2}{18} \ln \rho - \frac{20\rho^2}{3} \ln^2 \rho + \frac{2}{3} \zeta(3) + 2\rho^2 \zeta(3);$	
$g_2 = -\frac{385}{162} - \frac{41\pi^2}{162} - \frac{28\pi^2 \rho}{3} - \frac{89873\rho^2}{5184} - \frac{997\pi^2 \rho^2}{324} - \frac{1961\rho^2}{72} \ln \rho + 14\rho^2 \ln^2 \rho - \frac{5}{2} \zeta(3) - \frac{16\rho^2}{3} \zeta(3);$	
$g_3 = \frac{2691761}{202500} + \frac{3037\pi^2}{1350} + 24\pi^2 \rho + \frac{655429\rho^2}{97200} + \frac{2359\pi^2 \rho^2}{324} + \frac{6943\rho^2}{360} \ln \rho - 36\rho^2 \ln^2 \rho + \frac{42}{5} \zeta(3) + 15\rho^2 \zeta(3);$	
$p_0 = -\frac{343277101}{45000} - \frac{33156604927\rho^2}{583200} + \pi^2 \left(-\frac{615427}{4050} + \frac{6776\rho}{3} + \frac{763121\rho^2}{972} \right) - \frac{4\pi^4}{135} (7817 + 3212\rho^2) + \left(-\frac{7290521}{3240} + \frac{49622\pi^2}{27} - \frac{128\pi^4}{9} \right) \rho^2 \ln \rho + \left(-3388 - \frac{80\pi^2}{3} \right) \rho^2 \ln^2 \rho + \left(25642 + \frac{1515724\rho^2}{27} - 128\pi^2 \rho^2 - 160\rho^2 \ln \rho \right) \zeta(3) - \frac{1280}{3} \rho^2 \zeta(5);$	
$p_1 = \frac{89280434843}{972000} + \frac{248834878697\rho^2}{388800} - \frac{1}{324} \pi^2 (-533001 + 9110736\rho + 3110417\rho^2) + \frac{2}{135} \pi^4 (180247 + 73530\rho^2) + \left(\frac{11101973}{1080} - \frac{193400\pi^2}{9} + \frac{320\pi^4}{3} \right) \rho^2 \ln \rho + \frac{2}{3} (63269 + 300\pi^2) \rho^2 \ln^2 \rho + \frac{1}{45} (-13410977 + 100 (-292301 + 432\pi^2) \rho^2 + 54000\rho^2 \ln \rho) \zeta(3) + 3200\rho^2 \zeta(5);$	
$p_2 = -\frac{6209532853}{27000} - \frac{29997466847\rho^2}{19440} + \pi^2 \left(-\frac{114521}{30} + 71840\rho + \frac{1970140\rho^2}{81} \right) - \frac{4}{9} \pi^4 (14685 + 6032\rho^2) + \frac{1}{54} (190613 - 2847360\pi^2 + 11520\pi^4) \rho^2 \ln \rho - 80 (1347 + 5\pi^2) \rho^2 \ln^2 \rho + \frac{10}{9} (-658509 + (-1431463 + 1728\pi^2) \rho^2 + 2160\rho^2 \ln \rho) \zeta(3) - 6400\rho^2 \zeta(5);$	
$p_3 = \frac{49726331179}{324000} + \frac{7324831423\rho^2}{7290} + \pi^2 \left(\frac{3897971}{1620} - \frac{145880\rho}{3} - \frac{3977785\rho^2}{243} \right) + \frac{14}{27} \pi^4 (8269 + 3419\rho^2) + \frac{7}{81} (-81551 - 401520\pi^2 + 1440\pi^4) \rho^2 \ln \rho + \frac{140}{3} (1563 + 5\pi^2) \rho^2 \ln^2 \rho + \frac{35}{27} (-371889 + 16 (-50437 + 54\pi^2) \rho^2 + 1080\rho^2 \ln \rho) \zeta(3) + \frac{11200}{3} \rho^2 \zeta(5);$	

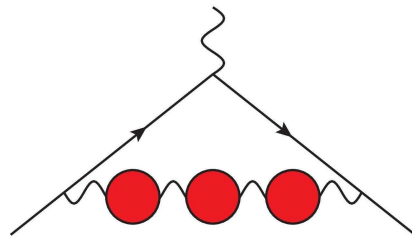
Table 3: The coefficients $g_i^{(6bl)}$, $h_i^{(6bl)}$, $j_i^{(6bl)}$, $p_i^{(6bl)}$ ($i = 0, 1, 2, 3$). The superscript (6bl) has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \text{Li}_4(1/2)$.

NNLO integrands 6a 6b 6bll



Plot of NNLO integrands 6a 6b 6bll

Huge, almost complete cancellations between positive and negative parts of integrands

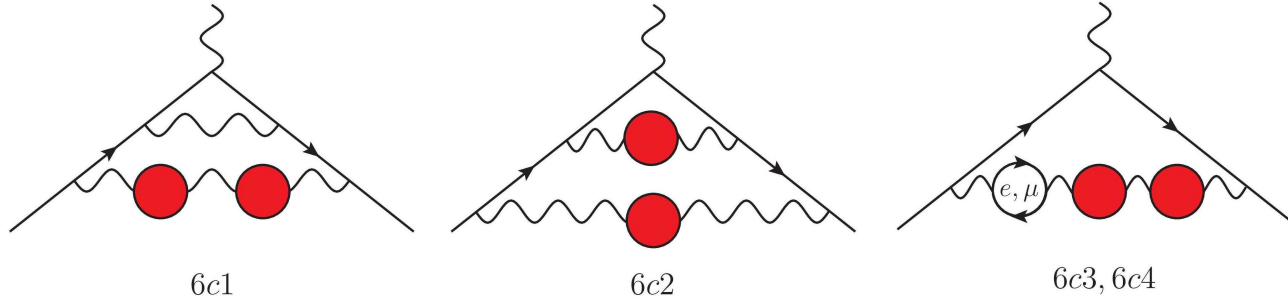


6d

$$a_{\mu}^{HVP}(\text{NNLO}; 6d) = \frac{\alpha}{\pi} \int_0^1 dx \kappa^{(2)}(x) [\Delta\alpha_{\text{had}}(t(x))]^3.$$

$$a_{\mu}^{HVP}(\text{NNLO}; 6d) = +0.005 \times 10^{-11}$$

very small contribution



$$a_{\mu}^{HVP}(\text{NNLO}; 6c) = a_{\mu}^{HVP}(\text{NNLO}; 6c1) + a_{\mu}^{HVP}(\text{NNLO}; 6c2) + a_{\mu}^{HVP}(\text{NNLO}; 6c3) + a_{\mu}^{HVP}(\text{NNLO}; 6c4)$$

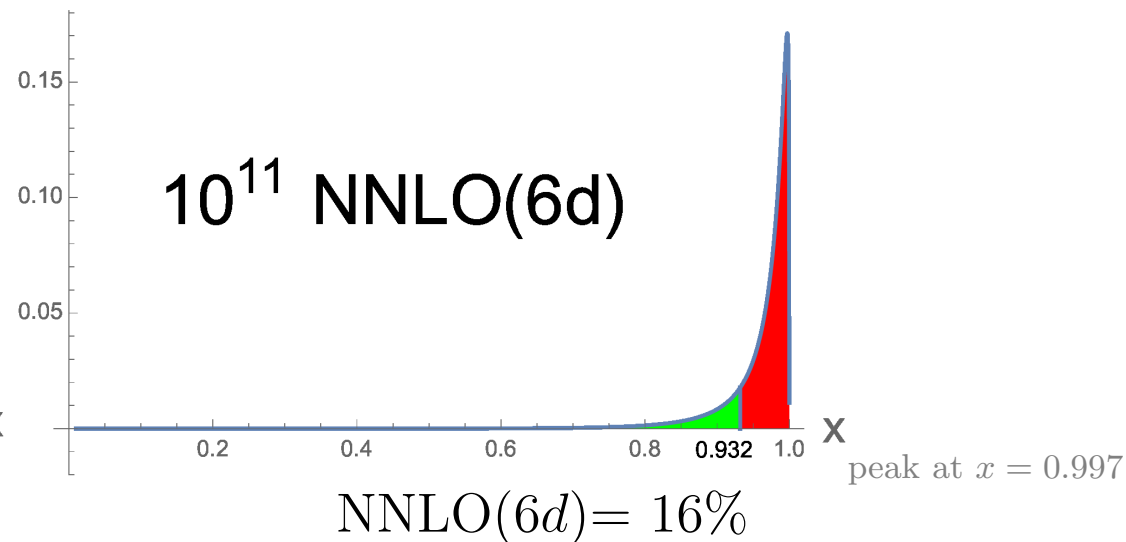
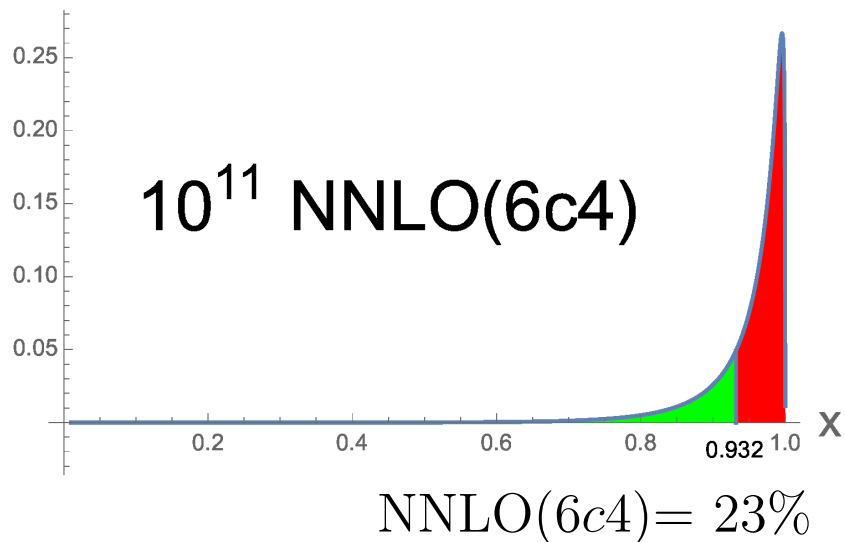
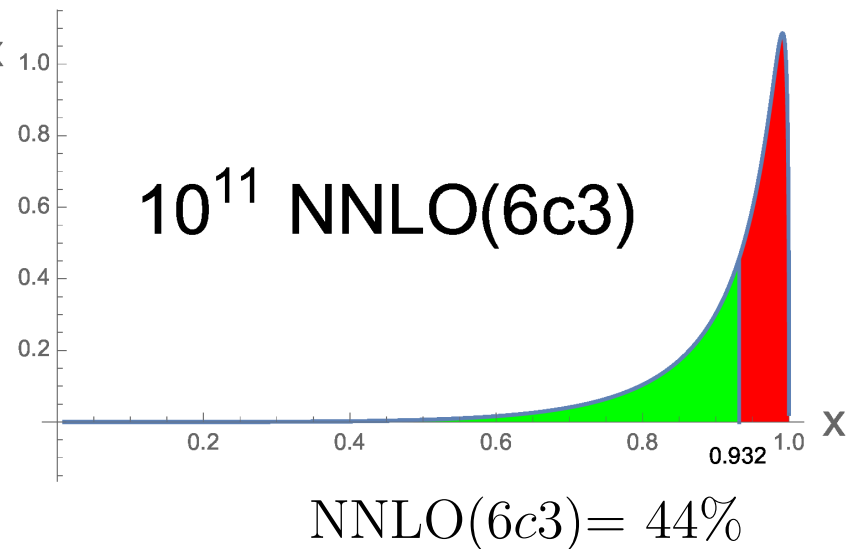
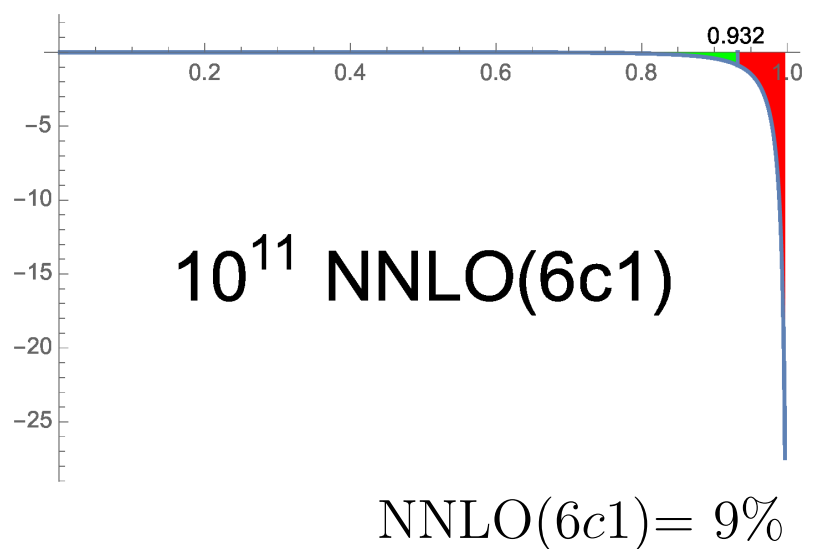
$$a_{\mu}^{HVP}(\text{NNLO}; 6c1) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx \left[\kappa^{(4)}(x) - \frac{2\pi}{\alpha} \kappa^{(2)}(x) \Delta\alpha_{\mu}^{(2)}(t(x)) \right] [\Delta\alpha_{\text{had}}(t(x))]^2 \quad \begin{array}{l} 6c4 \text{ separated} \\ \text{multiplicity}=3 \end{array}$$

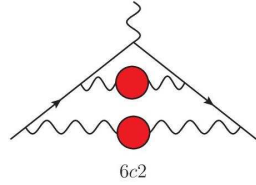
$$a_{\mu}^{HVP}(\text{NNLO}; 6c3) = \frac{3\alpha}{\pi} \int_0^1 dx \kappa^{(2)}(x) [\Delta\alpha_{\text{had}}(t(x))]^2 \Delta\alpha_e^{(2)}(t(x)).$$

$$a_{\mu}^{HVP}(\text{NNLO}; 6c4) = \frac{3\alpha}{\pi} \int_0^1 dx \kappa^{(2)}(x) [\Delta\alpha_{\text{had}}(t(x))]^2 \Delta\alpha_{\mu}^{(2)}(t(x)).$$

$$a_{\mu}^{HVP}(6c1) = -5 \times 10^{-12}, \quad a_{\mu}^{HVP}(6c3) = 0.9 \times 10^{-12}, \quad a_{\mu}^{HVP}(6c4) = 0.1 \times 10^{-12} \quad \text{6c2 ?}$$

NNLO integrands 6c1 6c3 6c4 6d





This class requires *double* integrals

$$a_{\mu}^{HVP}(\text{NNLO}; 6c2) = \frac{\alpha^2}{\pi^4} \int_{s_0}^{\infty} \frac{ds}{s} \int_{s_0}^{\infty} \frac{ds'}{s'} K^{(6c2)}(s/m^2, s'/m^2) \text{Im}\Pi_{\text{had}}(s) \text{Im}\Pi_{\text{had}}(s').$$

$$a_{\mu}^{HVP}(\text{NNLO}; 6c2) = \left(\frac{\alpha}{\pi}\right)^2 \int_{x_{\mu}}^1 dx \int_{x_{\mu}}^1 dx' \bar{\kappa}^{(6c2)}(x, x') \Delta\alpha_{\text{had}}(t(x)) \Delta\alpha_{\text{had}}(t(x')),$$

$\bar{\kappa}^{(6c2)}(x, x')$ space-like bidimensional kernel, $x_{\mu} < \{x, x'\} < 1$

$$\bar{\kappa}^{(6c2)}(x, x') = \frac{2-x}{x^3} \frac{2-x'}{x'^3} G^{(6c2)}\left(\frac{1-x}{x^2}, \frac{1-x'}{x'^2}\right)$$

From the leading terms of the known asymptotic expansion of $K^{(6c2)}(s/m^2, s'/m^2)$:

$s/s' \ll 1$ or $s/s' \approx 1$ or $s/s' \gg 1$ and $s, s' \gg m^2$ we get the approximated space-like kernel

$$G^{(6c2)}(\xi, \xi') = \frac{1855 - 188\pi^2}{4(32\pi^2 - 315)} \frac{\min(\xi, \xi')}{\max(\xi, \xi')^2} + \frac{988\pi^2 - 9765}{4(32\pi^2 - 315)} \frac{\min(\xi, \xi')^2}{\max(\xi, \xi')^3} + \frac{6(435 - 44\pi^2)}{32\pi^2 - 315} \frac{\min(\xi, \xi')^3}{\max(\xi, \xi')^4}$$

Contribution of 6c2 class is $a_{\mu}^{HVP}(6c2) = -1.8 \times 10^{-12}$

The uncertainty of this leading order approximation is estimated to be $\sim 10^{-13}$

of the diagram contribution NNLO(6c2) scanned by MUonE= 6%

SUMMARY

class	$10^{11}a_\mu(\text{diagram})$	$10^{11}a_\mu(\text{MUonE})$	% of the diagram contribution
LO(2)	6930	5828	84.1%
NLO(4a)	-207	-85	41.3%
NLO(4b)	+106	+87	82.0%
NLO(4c)	+3.4	+1.6	48.6%
NNLO(6a)	+8.1	+1.2	15%
NNLO(6b)	-4.0	-0.6	15%
NNLO(6bll)	+9.1	+3.5	38%
NNLO(6c1)	-0.50	-0.04	9%
NNLO(6c2)	-0.019	-0.001	6%
NNLO(6c3)	+0.09	+0.04	44%
NNLO(6c4)	+0.011	+0.002	23%
NNLO(6d)	+0.005	+0.0008	16%

Conclusions

- LO kernel: MUonE scans directly 84% of the contribution
- NLO kernels are known exactly;
- MUonE scans 41%, 82%, and 49% of the integrals of NLO (4a), (4b) and (4c), respectively.
- Despite the lack of analytical expressions, we are able to get approximated space-like NNLO kernels from the first terms of the asymptotic expansions. of the main three NNLO classes (6a), (6b) and (6bll); due to the peaking of integrands at $x \approx 1$, MUonE scans 15%, 15%, and 38% of the contributions (6a), (6b) and (6bll), respectively.
- For one set (6c2) containing two HVP insertions on *different* photon lines, we worked out a *bidimensional* approximated space-like kernel.
- The precision of the contributions of all the approximated space-like kernels obtained is at the level of 10^{-13} . Other classes are very small and are scanned by MUonE 6-15%.

THE END

The End