MUonE at N3LO

Marco Bonetti

Karlsruher Institut für Technologie

Muon Precision Physics 2024

Karlsruher Institut für Technologie

In collaboration with W. J. Torres Bobadilla

[WORK IN PROGRESS]

Muon $g - 2$

[2402.15410] [LATTICE2021.005]

A game of precision

- The SM is not the utimate answer
- At present no new particles

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$$a_\mu^{\text{Exp}} = 116\,592\,059(22) \times 10^{-11} \quad \Leftrightarrow \quad 0.19\,\text{ppm}$$

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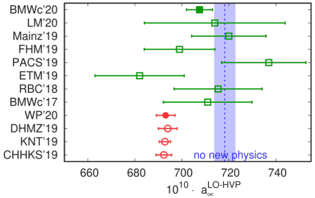

The MUonE project [1504.02228] [1609.08987] [1907.01574]

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- N3LO, full m_μ & remassified m_e
- NNLO, full m_e & m_μ

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N by N

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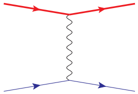

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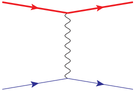
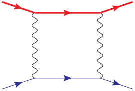
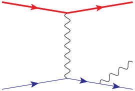


NLO





Dimensional regularization: singularities as poles in $\epsilon = (D-4)/2$

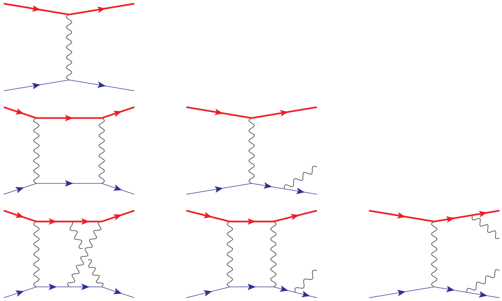
- UV: $\epsilon^{-\text{loop}}$ $\quad\epsilon^{-1}$
- IR: $\epsilon^{-2\ \text{emissions}}$ (mitigated by remassification) $\quad\epsilon^{-2}$

N by N

LO

NLO $\left[\epsilon^2\right]$

NNLO



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NLO $[\epsilon^4]$

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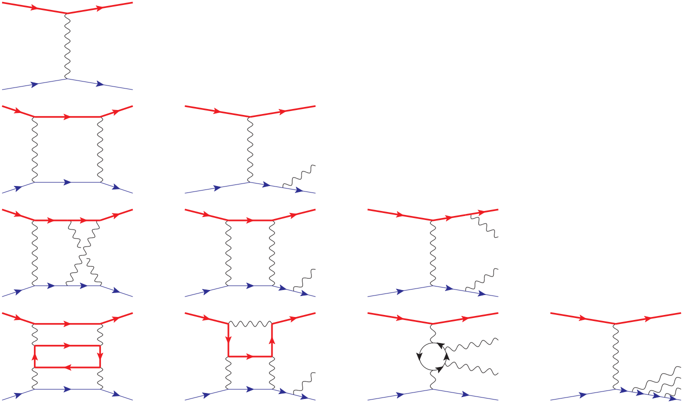
N^3LO



Dimensional regularization: singularities as poles in $\epsilon = (4 - D)/2$

- UV: $\epsilon^{-\text{loop}}$ $\qquad\qquad$ ϵ^{-3}
- IR: $\epsilon^{-2\ \text{emissions}}$ (mitigated by remassification) $\qquad$ ϵ^{-6}

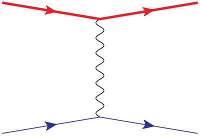
Pure virtual contributions [1712.08075]

Tensor structures

Pure virtual contributions

[1712.08075]

Tensor structures



0-loop $\Rightarrow$ $\left[\gamma\right]$ $\left[\gamma\right]$

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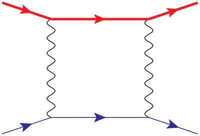

$$\text{0-loop} \Rightarrow \quad [\gamma]\ [\gamma]$$

$$\text{1-loop} \Rightarrow \quad [\gamma\gamma\gamma]\ [\gamma\gamma\gamma]$$

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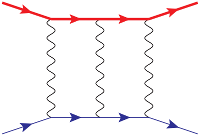

0-loop ⇒ $\color{blue}{[\gamma]}$ $\color{red}{[\gamma]}$

1-loop ⇒ $\color{blue}{[\gamma\gamma\gamma]}$ $\color{red}{[\gamma\gamma\gamma]}$

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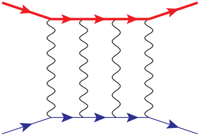

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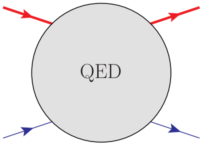
Relevant tensor structures



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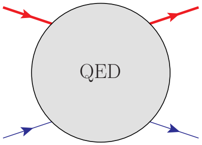

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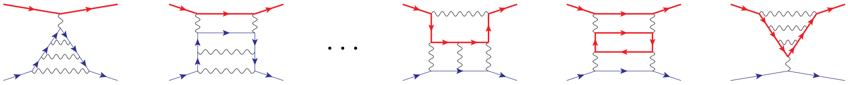
QED Abelian: partition by Q_μ



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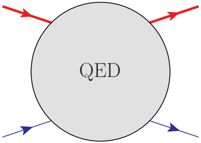

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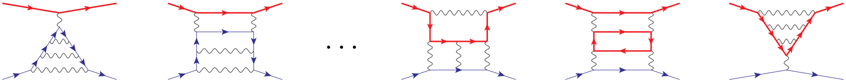

[RADCOR2009.038] [2207.00027]

Constants —— $\mathrm{d}\log\sqrt{P}$ —— —— Elliptic ——

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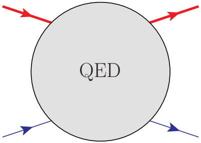

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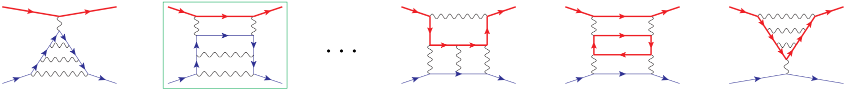

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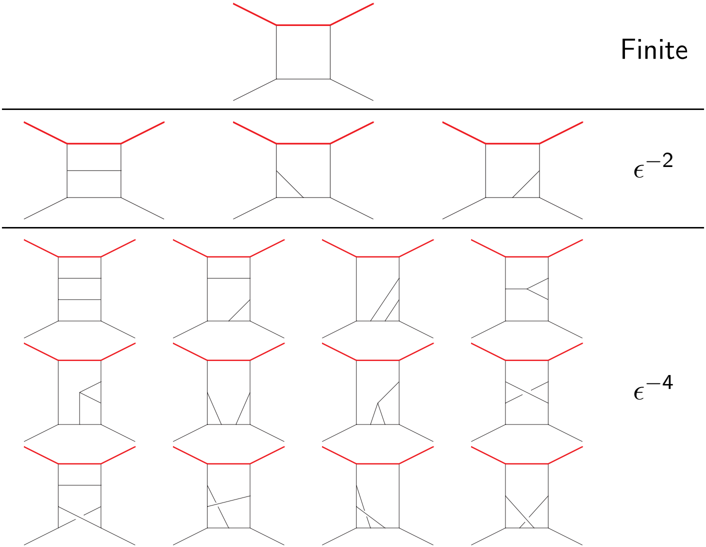

Master Integrals

1L 2 diagrams   2L 12 diagrams   3L 108 diagrams

Form factors sums of $\mathcal{O}\left(10^5\right)$ scalar integrals

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Master Integrals

Basis of loop integrals for the amplitude

1L 5 MIs 2L 23 [3] MIs 3L 345 [5] MIs

The roadmap

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Amplitude construction

- `qgraf`: Feynman diagram generation
- `Reduze`: sector mapping
- `FORM`: Dirac algebra, diagram summation
- `kira`: symbolic reduction to Master Integrals

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- Differential Equations

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Evaluation of the Master Integrals

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Finite remainder

- UV & IR subtraction
- Refinement of the form factors
- Helicity amplitudes, $|\mathcal{M}|^2$

Differential equations for Master Integrals

1. MIs are functions of kinematics

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Closed system of linear PDEs

$$\frac{\partial J(x,\epsilon)}{\partial x_i} = A_i(x,\epsilon) J(x,\epsilon)$$

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Canonical form

$$A(\mathrm{x},\epsilon) = \epsilon \sum_a B_a \, \mathrm{d}\log R_a(\mathrm{x})$$

- ϵ-homogeneous
- B_a contain rational numbers only
- $\mathrm{d}\log$ of algebraic functions in the kinematics

$$m^2 \qquad s \qquad t \qquad u \qquad \frac{s-\sqrt{s(s-4m^2)}}{s+\sqrt{s(s-4m^2)}}$$

$$m^4 - tu \qquad 4m^2 - s \qquad m^2 - t \qquad m^2 - u \qquad \frac{u-t-\sqrt{s(s-4m^2)}}{u-t+\sqrt{s(s-4m^2)}}$$

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Uniform weight

$$F(x,\varepsilon) = F_0^{(0)} + \varepsilon \left[\int_x \mathcal{B}(\xi_1)\,\mathrm{d}\xi_1\; F_0^{(0)} + F_0^{(1)}\right] + \\ + \varepsilon^2 \left[\int_x \mathcal{B}(\xi_1) \int_{\xi_1} \mathcal{B}(\xi_2)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_1\; F_0^{(0)} + \int_x \mathcal{B}(\xi_2)\,\mathrm{d}\xi_2 F_0^{(1)} + F_0^{(2)}\right] + \dots$$

- Only specific functions and constants at each order in ϵ
- Singularities and discontinuities from kernels
- Analytic cancellations in the amplitude

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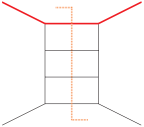
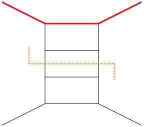
Thresholds & expansion-by-regions



$\Rightarrow \quad s = m^2$



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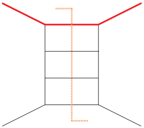

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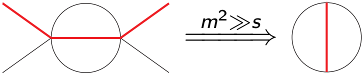

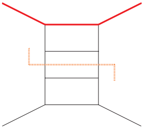

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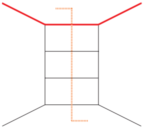

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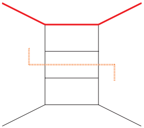

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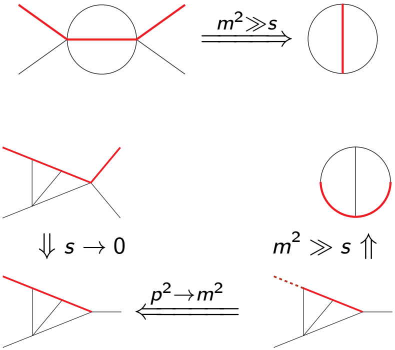

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Power series expansion

- Freeze integration path, expand either at DEs or at direct integration
- Code implementation or expanded amplitue

Wrap-up

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HVP to clarify

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HVP to clarify $\Leftarrow$ MUonE for independent result

Wrap-up

HVP to clarify $\Leftarrow$ MUonE for independent result $\Leftarrow$ Theory beyond NNLO

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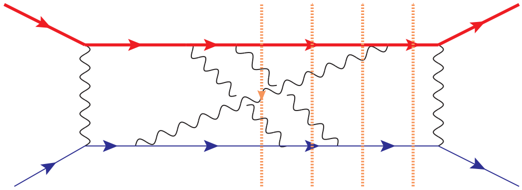
MUonE @ N3LO

- The goal



Wrap-up

HVP to clarify $\Leftarrow$ MUonE for independent result $\Leftarrow$ Theory beyond NNLO

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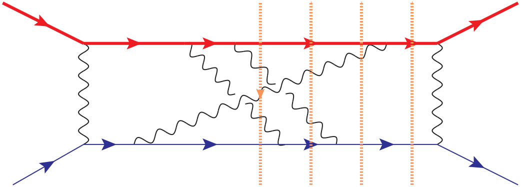

- Interesting math
- Chen iterated integrals
- Elliptic sectors
- N3LO UV & IR

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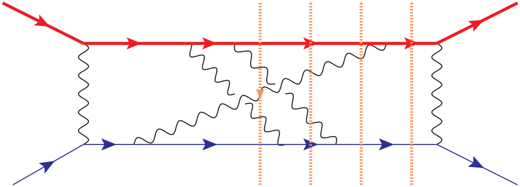

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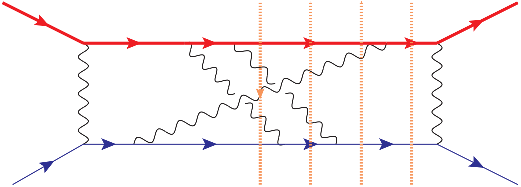

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Where do we stand now?

3-loop, one massive line (GPLs, NNLO complexity)

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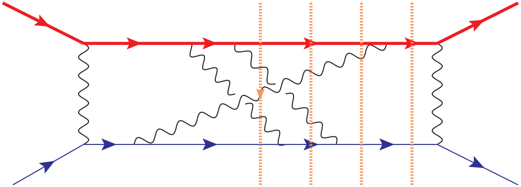

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STAY TUNED!

Thank you for your attention