High-order kernels in spacelike region

Stefano Laporta

Dipartimento di Fisica e Astronomia, Università di Padova, Italy

Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Padova, Italy

Stefano.Laporta@pd.infn.it

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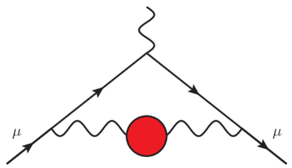


Leading order (LO) hadronic vacuum polarization contribution to muon g-2.

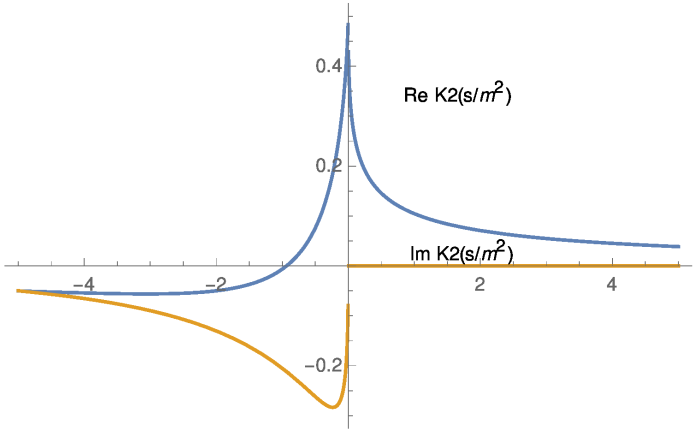
timelike dispersive integral spacelike dispersive integral

$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi^2} \int_{s_0=m^2_{\pi^0}}^{\infty} \frac{ds}{s} K^{(2)}(s/m_\mu^2) {\rm Im}\Pi(s) = -\frac{\alpha}{\pi^2} \int_{-\infty}^{0} \frac{dt}{t} \Pi(t) {\rm Im} K^{(2)}(t/m_\mu^2) = 6931(40) \times 10^{-11} \ ({\rm WP20})$$

$K^{(2)}(s/m_\mu^2)$: 1-loop QED g-2 contribution with a massive photon of mass $\sqrt{s}$

$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right) \ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}} \left(z - 2z^2 + \frac{z^3}{2}\right)$$

$${\rm Im} K^{(2)}(z + i\epsilon) = \pi \theta(-z) \left[\frac{z^2}{2} - z + \frac{z - 2z^2 + \frac{z^3}{2}}{\sqrt{z(z-4)}}\right] \qquad y(z) = \frac{z - \sqrt{z(z-4)}}{z + \sqrt{z(z-4)}}$$

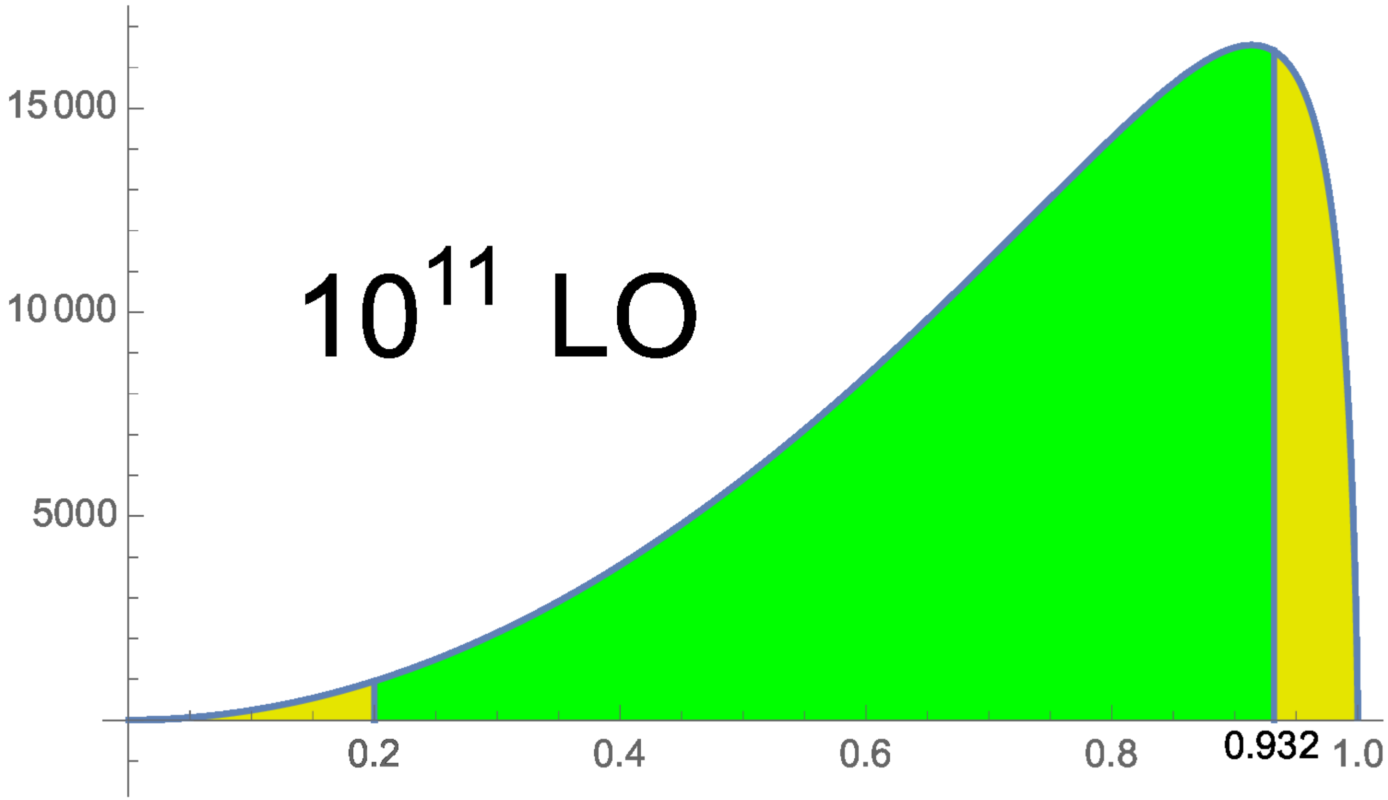


changing variable in the dispersive integral $t \to x(y(t/m_\mu^2)) = 1 + 1/y(t/m_\mu^2)$

$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi} \int_0^1 dx \ \kappa^{(2)}(x) \Delta\alpha_{\rm had}(t(x))$$

Lautrup, Peterman,deRafael 1972,Carloni Passera Trentadue Venanzoni 2015

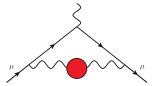
$$\boxed{\kappa^{(2)}(x) = 1 - x} \qquad \Delta\alpha_{\rm had}(t) = -\Pi(t) \qquad t(x) = m_\mu^2 \frac{x^2}{x-1}$$



Plot of spacelike LO integrand $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\rm had}(t(x))$ peak at $x = 0.914$

- with $E_\mu = 150 GeV$ MUonE will directly scan the region $0.2 < x < 0.932$
- Green=LO directly scanned by MUonE= 84% of $a_\mu^{\rm HVP}({\rm LO})$ 84 → 99% *alternative LO approach*

Ignatov, Pilato, Teubner and Venanzoni, *Phys.Lett.* **B** 848 (2024) 138344, arXiv:2309.14205



$$a_\mu^{\rm HVP}({\rm LO}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dT\, G(T)\, \tilde{K}^{(2)}(T)$$

- $G(T)$ correlator of e.m. currents $\leftarrow$ lattice
- $K^{(2)}(s) \xrightarrow{\text{Fourier transform}} \tilde{K}^{(2)}(T)$ LO time-kernel
- T Euclidean time

(Bernecker Meyer 2011)

$$\tilde{K}^{(2)}(T) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega} \left[\frac{1}{\pi}\frac{{\rm Im}K^{(2)}(-\omega^2/m_\mu^2)}{-\omega^2}\right]\left[\omega^2T^2 - 4\sin^2\left(\frac{\omega T}{2}\right)\right]$$

${\rm Im}K^{(2)}(q^2)$ LO space-like kernel

Analytical integration possible!: $\hat{T} = m_\mu T$ adimensional time $(\hat{T}=1 \;\rightarrow\; T = 1.86{\rm fm})$

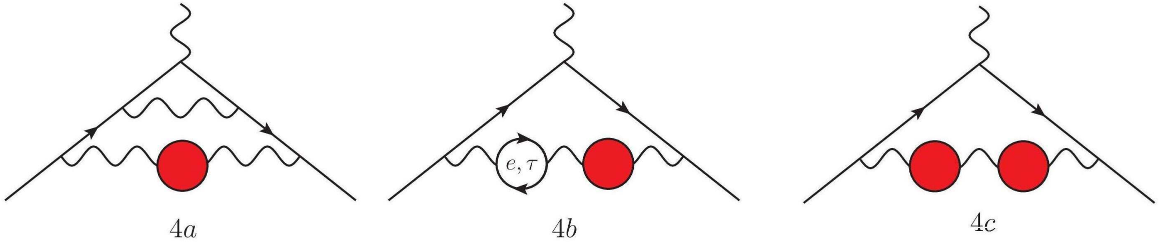
$$\frac{m_\mu^2}{8\pi^2}\tilde{K}^{(2)}(T) = \frac{1}{4}\overbrace{G^{2,1}_{1,3}\left(\begin{matrix}\frac{3}{2}\\ 0,1,\frac{1}{2}\end{matrix}\middle|\,\hat{T}^2\right)}^{\text{Meijer G-function}} + \frac{\hat{T}^2}{4} + \frac{1}{\hat{T}^2} + 2(\ln\hat{T}+\gamma) - \frac{2}{\hat{T}}K_1(2\hat{T}) - \frac{1}{2}$$

(Della Morte et al 2017)

$$= -\pi T^2\overbrace{\left(\mathbf{L}_{-1}(2\hat{T})K_0(2\hat{T}) + \mathbf{L}_0(2\hat{T})K_1(2\hat{T})\right)}^{\text{Struve Bessel functions}} + \frac{\hat{T}^2}{4} + \frac{1}{\hat{T}^2} - \left(\frac{2}{\hat{T}} + \hat{T}\right)K_1(2\hat{T}) + 2(\ln\hat{T}+\gamma) - \frac{1}{2}$$

(E.Balzani, S.L, M.Passera 2023)

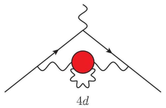
$$\frac{m_\mu^2}{8\pi^2}\tilde{K}^{(2)}(T) = \begin{cases} \frac{\hat{T}^4}{72} + \frac{(120(\ln\hat{T}+\gamma)-169}{43200}\hat{T}^6 + \dots & \hat{T} \ll 1 \\ \frac{T^2}{4} - \frac{\pi\hat{T}}{2} + 2(\ln\hat{T}+\gamma) - \frac{1}{2} + \frac{1}{\hat{T}^2} \underbrace{+\sqrt{\frac{\pi}{\hat{T}}}e^{-2\hat{T}}\left[-\frac{1}{4} - \frac{55}{64\hat{T}} - \frac{729}{2048\hat{T}^2} + \frac{10515}{32768\hat{T}^3} + \dots\right]}_{\text{exponentially suppressed}} & \hat{T} \gg 1 \end{cases}$$



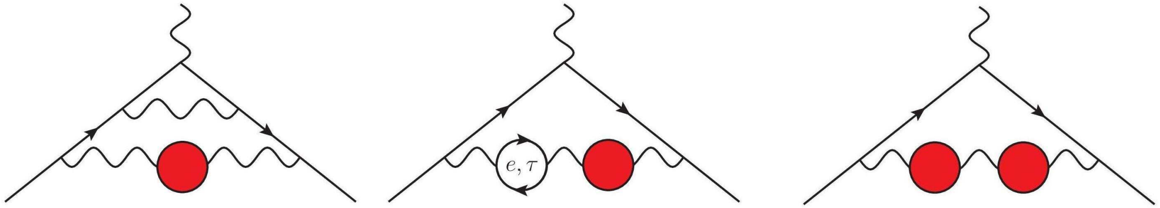
- Class a: 1 HVP insertion in one photon line of all 2-loop QED vertex diagrams
- Class b: 1 HVP insertion in the photon line of all 2-loop QED vertex with one electron vacuum polarization
- Class c: 2 HVP insertion in the 1-loop QED vertex diagram

$$a_\mu^{\rm HVP}({\rm NLO}; 4a) = -209.0 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; 4b) = +106.8 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; 4c) = +3.5 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; total) = -98.7(9) \times 10^{-11}$$

(Krause 1996, Hagiwara Liao Martin Nomura Toebner 2011, Kurz Liu Marquard Steinhauser 2014)



HVP insertion with internal corrections already incorporated in LO



timelike and spacelike integral:

$$a_\mu^{\rm HVP}({\rm NLO};4a) = \frac{\alpha^2}{\pi^3}\int_{s_0}^{\infty}\frac{ds}{s}\,2K^{(4)}(s/m_\mu^2)\,{\rm Im}\Pi(s) = -\frac{\alpha^2}{\pi^3}\int_{-\infty}^{0}\frac{dt}{t}\Pi(t)\,{\rm Im}2K^{(4)}(t/m_\mu^2)$$

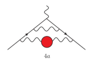
$2K^{(4)}(s/m_\mu^2)$: 2-loop QED g-2 contribution from diagrams with one massive photon of mass $\sqrt{s}$ and one massless photon (factor 2 due to normalization chosen)

$$a_\mu^{\rm HVP}({\rm NLO};4b) = \frac{\alpha}{\pi}\int_0^1 dx\,\kappa^{(2)}(x)\Delta\alpha_{\rm had}(t(x))\;2\left(\Delta\alpha_e^{(2)}(t(x)) + \;\Delta\alpha_\tau^{(2)}(t(x))\right)$$

$$a_\mu^{\rm HVP}({\rm NLO};4c) = \frac{\alpha}{\pi}\int_0^1 dx\,\kappa^{(2)}(x)\,(\Delta\alpha_{\rm had}(t(x)))^2$$

$$-\Delta\alpha_{\rm l}(t) = \Pi_l^{(2)}(t) = \left(\frac{\alpha}{\pi}\right)\left[\frac{8}{9} - \frac{\beta_l^2}{3} + \beta_l\left(\frac{1}{2} - \frac{\beta_l^2}{6}\right)\ln\frac{\beta_l - 1}{\beta_l + 1}\right]\;,\quad \beta_l = \sqrt{1 - 4m_l^2/t}$$

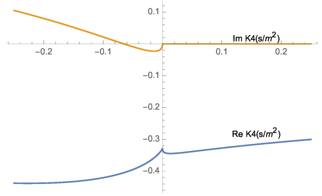
Π_l renormalized one-loop QED vacuum polarization function

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4a) = \left(\frac{\alpha}{\pi}\right)^2 \int\limits_0^1 dx\, \kappa^{(4)}(x)\Delta\alpha_{\mathrm{had}}(t(x))$$

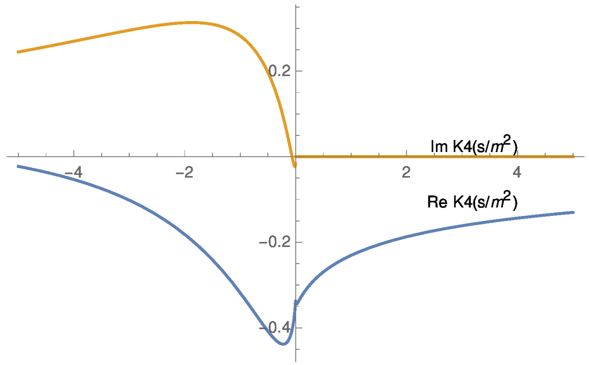
Space-like NLO kernel $\kappa^{(4)}(x)$

$$\kappa^{(4)}(x) = \frac{2(2-x)}{x(x-1)} F^{(4)}(x-1)$$



$$F^{(4)}(u) = \frac{-3u^4-5u^3-7u^2-5u-3}{6u^2}\left(2\mathrm{Li}_2(-u) + 4\mathrm{Li}_2(u) + \ln(-u)\ln\left((1-u)^2(u+1)\right)\right)$$
$$+ \frac{(u+1)\left(-u^3+7u^2+8u+6\right)}{12u^2}\ln(u+1) + \frac{\left(-7u^4-8u^3+8u+7\right)}{12u^2}\ln(1-u)$$
$$+ \frac{23u^6-37u^5+124u^4-86u^3-57u^2+99u+78}{72(u-1)^2u(u+1)}$$
$$+ \frac{12u^8-11u^7-78u^6+21u^5+4u^4-15u^3+13u+6}{12(u-1)^3u(u+1)^2}\ln(-u)$$

$$\mathrm{Im}K^{(4)}(z+i\epsilon) = \pi\theta(-z)F^{(4)}(1/y(z)) \qquad y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}} < -1$$

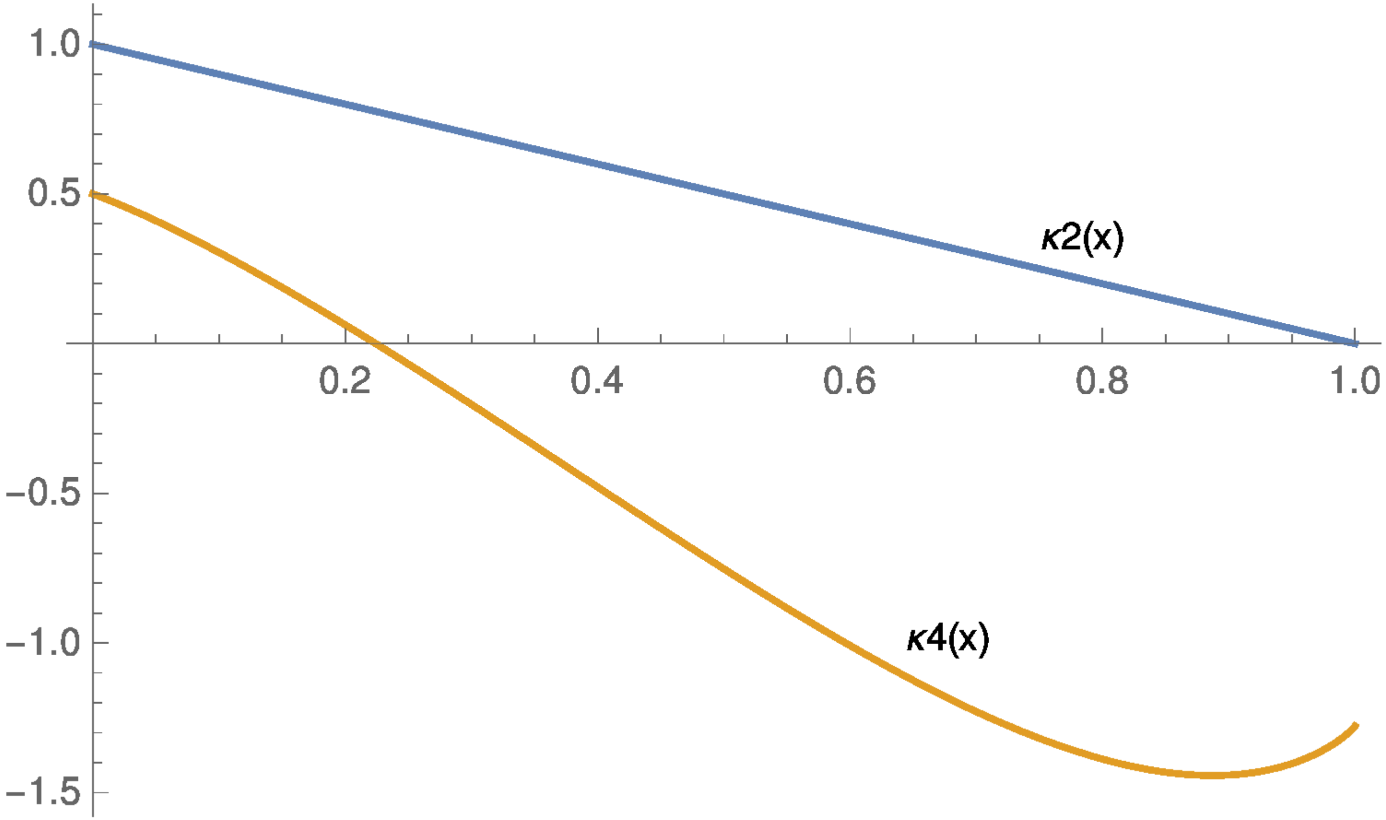


Balzani, S.L., Passera 2112.05704, Nesterenko 2112.05009.

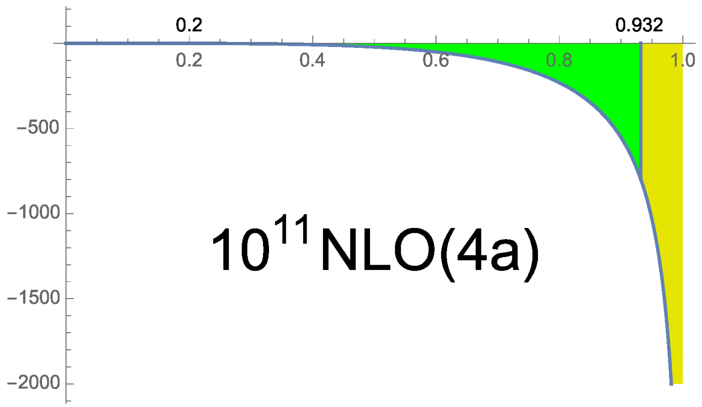
$$K^{(4)}(z) = \left(\frac{z^2}{2} - \frac{7z}{6} + \frac{1}{2}\right)\left[-3\mathrm{Li}_3(-y) - 6\mathrm{Li}_3(y) + 2\left(\mathrm{Li}_2(-y) + 2\mathrm{Li}_2(y)\right)\ln y + \frac{1}{2}\left(\ln^2 y + \pi^2\right)\ln(y+1) + \ln(1-y)\ln^2 y\right]$$
$$+ \frac{\left(-\frac{z^3}{6}+\frac{z^2}{4}-\frac{7z}{6}-\frac{4}{z-4}+\frac{13}{3}\right)\left(\mathrm{Li}_2(-y)+\frac{\ln^2 y}{4}+\frac{\pi^2}{12}\right)}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{12}+\frac{17z^2}{6}-2z\right)\left(\mathrm{Li}_2(y)-\frac{1}{4}\ln^2 y+\ln(1-y)\ln y-\frac{\pi^2}{6}\right)}{\sqrt{(z-4)z}}$$
$$+ \left(-\frac{29z^2}{96} + \frac{53z}{48} + \frac{2}{z-4} - \frac{1}{3z} + \frac{19}{24}\right)\ln^2 y + \frac{\left(\frac{23z^3}{144}-\frac{115z^2}{72}+\frac{127z}{36}-\frac{4}{3}\right)\ln y}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{48}+\frac{17z^2}{24}-\frac{z}{2}\right)\ln y\ln z}{\sqrt{(z-4)z}}$$
$$+ \frac{1}{6}\pi^2\left(-\frac{z^2}{2} + \frac{5z}{24} - \frac{2}{z} + \frac{9}{4}\right) + \frac{5}{96}z^2\ln^2 z + \left(\frac{23z^2}{144} - \frac{7z}{36} + \frac{1}{z-4} + \frac{19}{12}\right)\ln z + \frac{115z}{72} - \frac{139}{144}$$

Barbieri Remiddi 1975

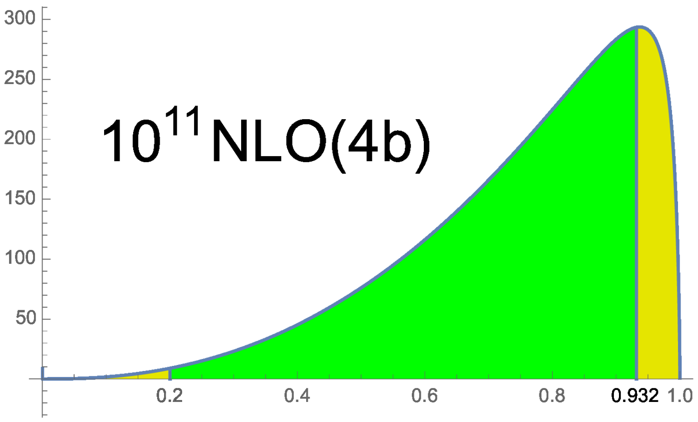
$K^{(4)}(0) = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta(3) = -0.328479$ 2-loop g-2 $\quad K^{(4)}(z \gg 1) \to \frac{1}{z}\left(-\frac{23\ln(z)}{36} - \frac{\pi^2}{3} + \frac{223}{54}\right)$



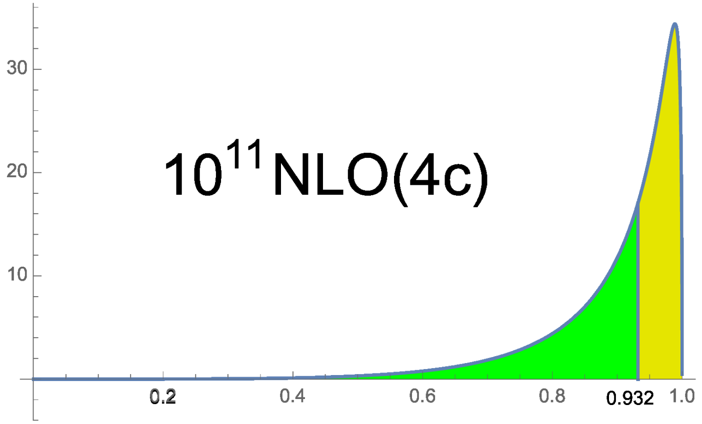
- $\kappa^{(4)}(x)$ changes sign at $x = 0.224212$ ($\to q^2 \approx -(23\text{MeV})^2$)
- $\kappa^{(4)}(1) = -\frac{23}{18}, \quad \kappa^{(4)}(0) = \frac{1}{2};$
- $\kappa^{(4)}(x)$ provides stronger weight a large $q^2 < 0$ ($x \to 1$) than $\kappa^{(2)}(x)$



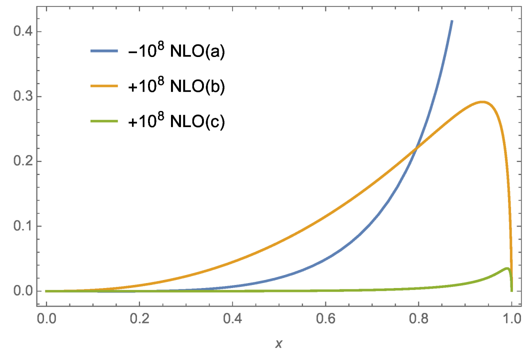
Plot of spacelike NLO($4a$) integrand $(\alpha/\pi)^2\kappa^{(4)}(x)\Delta\alpha_{\rm had}(t(x))$
logarithmic singularity for $x = 1$



Plot of spacelike NLO($4b$) integrand $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\rm had}(t(x))\ 2\left(\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x))\right)$
peak at $x = 0.936$



Plot of spacelike NLO($4c$) integrand $(\alpha/\pi)\kappa^{(2)}(x)\left(\Delta\alpha_{\rm had}(t(x))\right)^2$
peak at $x = 0.988$



the 3 NLO integrands at the same scale

- with $E_\mu = 150 GeV$ MUonE will directly scan the region $0.2 < x < 0.932$
- Green=directly scanned by MUonE: 41% of $a_\mu^{\rm HVP}({\rm NLO};4a)$, 82% of $a_\mu^{\rm HVP}({\rm NLO};4b)$, 49% of $a_\mu^{\rm HVP}({\rm NLO};4c)$
- $a_\mu^{\rm HVP}({\rm NLO})$: can we apply the alternative approach?

Ignatov, Pilato, Teubner and Venanzoni, *Phys.Lett.***B** 848 (2024) 138344, arXiv:2309.14205

- splitting the *timelike* integral in low and high-energy regions
- fit approximations $K_1(s)$, $\tilde{K}_1(s)$ to the timelike kernel $K(s)$ in both regions
- split the integral and express integrals of fitting functions with derivatives of $\Delta\alpha_h(t)$ at $t = 0$ (obtained from MUonE data) and contours integrals in the complex plane (obtained from pQCD).

$$a_\mu^{\text{HVP;NLO}} = a_\mu^{\text{HVP;NLO(I)}} + a_\mu^{\text{HVP;NLO(II)}} + a_\mu^{\text{HVP;NLO(III)}} + a_\mu^{\text{HVP;NLO(IV)}}$$

$$a_\mu^{\text{HVP;NLO(I)}} = -\left(\frac{\alpha}{\pi}\right)^{1+1} \sum \frac{c_n^{(\text{NLO})}}{n!} \frac{d^n}{d\,t^n} \Delta\alpha_{had}(t)\Big|_{t=0}$$

$$a_\mu^{\text{HVP;NLO(II)}} = -\left(\frac{\alpha}{\pi}\right)^{1+1} \frac{1}{2\pi i} \int\limits_{|s|=s_0} \frac{ds}{s} \left(K_1^{(\text{NLO})}(s) - \tilde{K}_1^{(\text{NLO})}(s)\right) \Pi_{had}(s)\Big|_{\text{pQCD}}$$

$$a_\mu^{\text{HVP;NLO(III)}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^{2+1} \int\limits_{s_{\text{th}}}^{s_0} \frac{ds}{s} \left(K^{(\text{NLO})}(s) - K_1^{(\text{NLO})}(s)\right) R(s)$$

$$a_\mu^{\text{HVP;NLO(IV)}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^{2+1} \int\limits_{s_0}^{\infty} \frac{ds}{s} \left(K^{(\text{NLO})}(s) - \tilde{K}_1^{(\text{NLO})}(s)\right) R(s)$$

$$K_1^{(\text{NLO})}(s) \approx c_0^{(\text{NLO})} s + \frac{c_1^{(\text{NLO})}}{s} + \frac{c_2^{(\text{NLO})}}{s^2} + \frac{c_3^{(\text{NLO})}}{s^3}\,, \qquad s_{\text{th}} \leq s \leq s_0$$

$$\tilde{K}_1^{(\text{NLO})}(s) \approx \qquad + \frac{\tilde{c}_1^{(\text{NLO})}}{s} + \frac{\tilde{c}_2^{(\text{NLO})}}{s^2} + \frac{\tilde{c}_3^{(\text{NLO})}}{s^3}\,, \qquad s \geq s_0$$

Minimization I (*least square fit*)

s_0	$(1.8\text{GeV})^2$	$(2.5\text{GeV})^2$	$(12\text{GeV})^2$
$a_\mu^{\text{HVP;LO(I)}} \cdot 10^{11}$	6868.0	6899.2	6944.7
$a_\mu^{\text{HVP;LO(II)}} \cdot 10^{11}$	58.8	36.2	2.9
$a_\mu^{\text{HVP;LO(III)}} \cdot 10^{11}$	4.1	-4.5	-16.7
$a_\mu^{\text{HVP;LO(IV)}} \cdot 10^{11}$	-0.011	0.005	$-1.3 \cdot 10^{-7}$
total	6930.9	6930.9	6930.9

$a_\mu^{\text{HVP;LO(II)}} \sim 1\%\ a_\mu^{\text{HVP;LO}}$ at $s_0 = (1.8\text{GeV})^2$

s_0	$(1.8\text{GeV})^2$	$(2.5\text{GeV})^2$	$(12\text{GeV})^2$
$a_\mu^{\text{HVP;NLO(4a)(I)}} \cdot 10^{11}$	-187.5	-194.8	-211.4
$a_\mu^{\text{HVP;NLO(4a)(II)}} \cdot 10^{11}$	-20.2	-14.8	-2.3
$a_\mu^{\text{HVP;NLO(4a)(III)}} \cdot 10^{11}$	-0.05	1.98	6.07
$a_\mu^{\text{HVP;NLO(4a)(IV)}} \cdot 10^{11}$	0.074	-0.082	$2.3 \cdot 10^{-4}$
total	-207.7	-207.7	-207.7

$a_\mu^{\text{HVP;NLO(4a)(II)}} \sim 10\%\ a_\mu^{\text{HVP(4a);NLO}}$ at $s_0 = (1.8\text{GeV})^2$

$a_\mu^{\text{HVP}}(\text{NLO}; 4a) : 41\% \to 90\%$ *(preliminary)*

(work in progress)

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO;4a}) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dT\, G(T)\, \tilde{K}^{(4)}(T)$$

- $G(T)$ correlator of e.m. currents $\leftarrow$ lattice
- $K^{(4)}(s) \xrightarrow{\text{Fourier transform}} \tilde{K}^{(4)}(T)$ NLO(4a) time-kernel
- T Euclidean time

$$\tilde{K}^{(4)}(T) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega} \left[\frac{2\, F^{(4)}(1/y(-\omega^2/m_\mu^2))}{-\omega^2}\right] \left[\omega^2 T^2 - 4\sin^2\left(\frac{\omega T}{2}\right)\right]$$

$F^{(4)}$: NLO(4a) space-like kernel

- integral with $(\omega T)^2$ analytically easy; integral with $\sin^2(\omega T/2)$ difficult.
- not able to integrate analytically all the terms of $F^{(4)}$
- $\rightarrow$ Series expansions
- small-T expansion is straightforward
- large-T expansion is asymptotic

splitting the ω-integral and expanding the factors

$$\frac{m_\mu^2}{16\pi^2}\tilde{K}^{(4)}(T) = \sum_{\substack{n\geq 4\\ n \text{ even}}} \frac{\hat{T}^n}{n!}\left(a_n + b_n\pi^2 + c_n\left(\ln(\hat{T})+\gamma\right) + d_n\left(\ln(\hat{T})+\gamma\right)^2\right)$$

$\mathbf{n}$	$\mathbf{a_n}$	$\mathbf{b_n}$	$\mathbf{c_n}$	$\mathbf{d_n}$
4	$\frac{317}{216}$	$-\frac{1}{3}$	$\frac{23}{18}$	0
6	$\frac{843829}{259200}$	$-\frac{371}{432}$	$\frac{877}{1080}$	$\frac{19}{36}$
8	$\frac{412181237}{5292000}$	$-\frac{233}{48}$	$-\frac{824603}{25200}$	$\frac{141}{20}$
10	$\frac{6272504689}{10584000}$	$-\frac{1165}{48}$	$-\frac{460711}{1680}$	$\frac{961}{20}$
12	$\frac{404220031035193}{121022748000}$	$-\frac{42443}{378}$	$-\frac{1359283213}{873180}$	$\frac{79342}{315}$
14	$\frac{14790819716039431}{890463974400}$	$-\frac{142931}{288}$	$-\frac{4138386457}{540540}$	$\frac{28243}{24}$
16	$\frac{38888413518277699}{503454631680}$	$-\frac{12895145}{6048}$	$-\frac{489120278261}{13970880}$	$\frac{2605993}{504}$
18	$\frac{3950633085365067019}{11462583132000}$	$-\frac{116506871}{12960}$	$-\frac{4589675124823}{29937600}$	$\frac{23642359}{1080}$
20	$\frac{36472186980263447757757 1}{243865691961091200}$	$-\frac{55559731}{1485}$	$-\frac{37593205363634911}{57616158600}$	$\frac{44767436}{495}$
22	$\frac{7739223928279394588224 9}{12165635426630400}$	$-\frac{610873921}{3960}$	$-\frac{26135521670035411}{9602693100}$	$\frac{121188929}{330}$
24	$\frac{273187709279653799136705 22297}{1024872666654481444800}$	$-\frac{19509636989}{30888}$	$-\frac{5138081420797732289}{459392837904}$	$\frac{3789385597}{2574}$
26	$\frac{44996849076816882871466 5100663}{407619810601214211 0000}$	$-\frac{5618399257}{2184}$	$-\frac{1581091180177381766 9}{348024877200}$	$\frac{151912159}{26}$
28	$\frac{25114629392949805515668 3549773}{554584776328182600000}$	$-\frac{678234361}{65}$	$-\frac{3787066553671821473}{20715766500}$	$\frac{1495034796}{65}$
30	$\frac{1007921174630176846435552241782691 68501}{5468055457076246304990720 0000}$	$-\frac{2551294690547}{60480}$	$-\frac{30599625762869165887553 3}{419236121304000}$	$\frac{64743309493}{720}$

Table 1: Coefficients of the expansion of $\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t)$ up to $\hat{t}^{30}$,

$$\frac{m_\mu^2}{16\pi^2}\tilde{K}^{(4)}(T) = \frac{\hat{T}^2}{2}\left(\tfrac{197}{144}+\tfrac{\pi^2}{12}-\tfrac{1}{2}\pi^2\ln 2+\tfrac{3}{4}\zeta(3)\right)-\frac{\pi\hat{T}}{8}+\left(\ln\hat{T}+\gamma\right)\left(1-\frac{5}{12\hat{T}^2}\right)-\frac{7\zeta(3)}{4}+\frac{7}{6}\pi^2\ln(2)-\frac{127\pi^2}{144}+\frac{653}{216}$$
$$+\pi\left[-\frac{1}{2\hat{T}}+\frac{277}{360\hat{T}^3}+\frac{313}{1120\hat{T}^5}+\frac{101}{128\hat{T}^7}+\frac{99175}{16896\hat{T}^9}+\frac{24332175}{292864\hat{T}^{11}}+\frac{202954815}{106496\hat{T}^{13}}+\frac{35848186605}{557056\hat{T}^{15}}+\ldots\right]$$
$$+\left(\frac{209}{180\hat{T}^2}-\frac{11}{280\hat{T}^4}-\frac{53}{252\hat{T}^6}-\frac{81}{22\hat{T}^8}-\frac{13158}{143\hat{T}^{10}}-\frac{41160}{13\hat{T}^{12}}-\frac{2477520}{17\hat{T}^{14}}-\frac{2813680800}{323\hat{T}^{16}}+\ldots\right)$$
$$+e^{-2\hat{T}}\left[\pi^2\left(-\frac{1}{4}-\frac{23}{24\hat{T}}-\frac{67}{48\hat{T}^2}+\frac{5}{24\hat{T}^3}+\frac{1}{16\hat{T}^4}-\frac{17}{16\hat{T}^5}+\frac{175}{32\hat{T}^6}-\frac{105}{4\hat{T}^7}+\frac{525}{4\hat{T}^8}-\frac{11235}{16\hat{T}^9}+\frac{129465}{32\hat{T}^{10}}-\frac{401625}{16\hat{T}^{11}}+\ldots\right)\right.$$
$$+\left(\log(\hat{T}/2)+\gamma\right)\sqrt{\pi}\left(\frac{1}{2\hat{T}^{3/2}}+\frac{169}{96\hat{T}^{5/2}}-\frac{877}{3072\hat{T}^{7/2}}-\frac{32549}{49152\hat{T}^{9/2}}+\frac{4571915}{1048576\hat{T}^{11/2}}-\frac{355430947}{16777216\hat{T}^{13/2}}+\ldots\right)$$
$$\left.+\sqrt{\pi}\left(\frac{95}{144\hat{T}^{1/2}}+\frac{5897}{2304\hat{T}^{3/2}}+\frac{204643}{368640\hat{T}^{5/2}}+\frac{136124953}{41287680\hat{T}^{7/2}}-\frac{13694145841}{2642411520\hat{T}^{9/2}}+\frac{633979991035}{93012885504\hat{T}^{11/2}}+\ldots\right)\right]$$

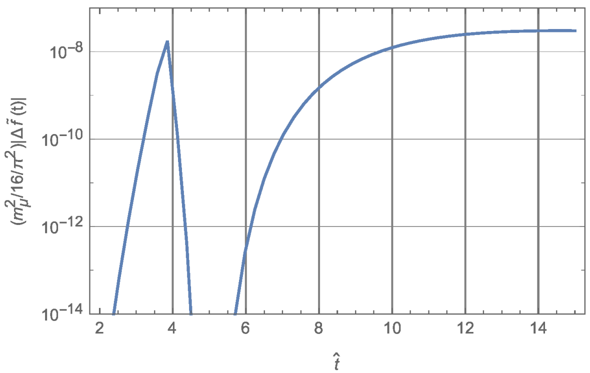
- series expansion consists of five different series
- all series expansions are asymptotic: factorial growth of coefficients
- there is a dominant part and exponentially suppressed part $\sim e^{-2\hat{T}}$
- numerically of limited use, as an asymptotic series needs truncation, : and the truncation error of the dominant part is $\sim e^{-2\hat{T}}$, which is the size of the exponentially suppressed part:

- The series converge for $\hat{T} \geq \hat{T}_0/2$, even for $\hat{T}=\infty$
- For each of the five series we get a Laplace integral representation through rotation in the $\omega-$ complex plane of the Fourier integral, from which we expand and obtain numerically the coefficients $a_n^{(b;x;y)}$

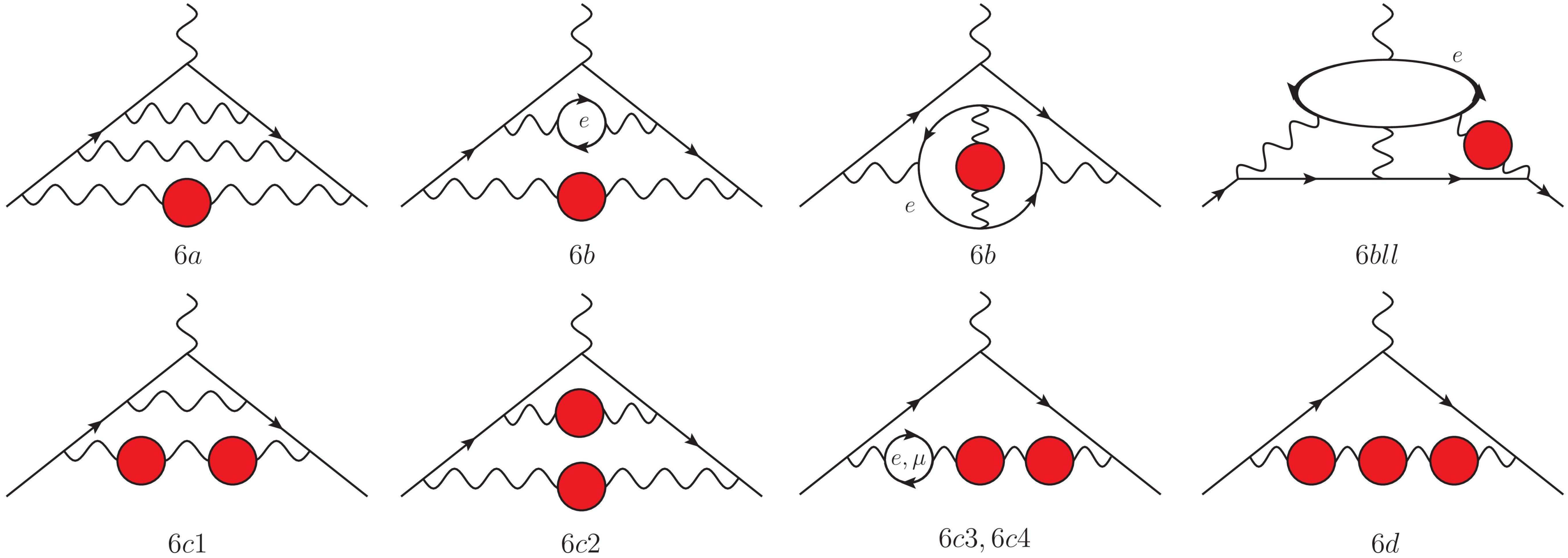
$$\frac{m_\mu^2}{16\pi^2}\tilde{K}^{(4)}(T) = \frac{\hat{T}^2}{2}\left(\frac{197}{144}+\frac{\pi^2}{12}-\frac{1}{2}\pi^2\ln 2+\frac{3}{4}\zeta(3)\right)-\frac{\pi\hat{T}}{8}+\left(\ln\hat{T}+\gamma\right)\left(1-\frac{5}{12\hat{T}^2}\right)-\frac{7\zeta(3)}{4}+\frac{7}{6}\pi^2\ln(2)-\frac{127\pi^2}{144}+\frac{653}{216}$$
$$+\frac{1}{\hat{T}}\sum_{n=0}^{\infty}a_n^{(b;1;1)}\left(\frac{\hat{T}_0^2}{\hat{T}^2}-1\right)^n+\frac{1}{\hat{T}^2}\sum_{n=0}^{\infty}a_n^{(b;1;2)}\left(\frac{\hat{T}_0^2}{\hat{T}^2}-1\right)^n+e^{-2\hat{T}}\sum_{n=0}^{\infty}a_n^{(b;2;1)}\left(\frac{\hat{T}_0}{\hat{T}}-1\right)^n$$
$$+\frac{e^{-2\hat{T}}}{\sqrt{\hat{T}}}\ln(\hat{T})\sum_{n=0}^{\infty}a_n^{(b;2;2)}\left(\frac{\hat{T}_0}{\hat{T}}-1\right)^n+\frac{e^{-2\hat{T}}}{\sqrt{\hat{T}}}\sum_{n=0}^{\infty}a_n^{(b;2;3)}\left(\frac{\hat{T}_0}{\hat{T}}-1\right)^n$$

n	$a_n^{(b;1,1)}$	$a_n^{(b;1,2)}$	$a_n^{(b;2,1)}$	$a_n^{(b;2,2)}$	$a_n^{(b;2,3)}$
0	−1.4724671380	1.1589872337	−4.8942765691	0.2973718753	2.1170734478
1	0.1002442629	−0.0022459376	−2.9475017651	0.4127862149	1.0364595246
2	0.0021557710	0.0008279191	−0.5075497783	0.1109534688	0.1101698869
3	0.0001282655	0.0007999410	0.0115794503	−0.0040980259	0.0167667530
4	−0.0001467432	−0.0006094594	−0.0013940058	0.0003899989	−0.0035236970
5	9.35581×10^{-6}	7.37693×10^{-6}	0.0001421294	−0.0000133805	0.0008586372
6	0.0000260037	0.0002711371	7.67679×10^{-6}	−0.00001764961	−0.0002257379
7	−0.0000189910	−0.0002551246	−0.00001492424	.000011742325	0.0000612688
8	6.93309×10^{-6}	0.0001291619	8.61706×10^{-6}	-5.92454×10^{-6}	−0.0000164422
9	3.18779×10^{-7}	−0.0000121615	-4.20065×10^{-6}	2.78837×10^{-6}	4.04750×10^{-6}
10	-2.93399×10^{-6}	−0.0000553459	1.95419×10^{-6}	-1.29025×10^{-6}	-7.17744×10^{-7}
11	2.98580×10^{-6}	0.0000760414	-9.00478×10^{-7}	5.98351×10^{-7}	-7.67136×10^{-8}
12	-2.08433×10^{-6}	−0.0000669985	4.17032×10^{-7}	-2.80343×10^{-7}	1.94188×10^{-7}

Table 2: Coefficients of the expansions in v of $\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t)$ up to v^{12} with $\hat{t}_0=5$,



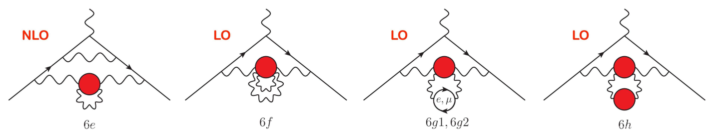
- Time-kernel for diagrams 4a, together with those for diagrams 4b and 4c were recently used in a preliminary lattice determination of the NLO HVP contributions to the $(g\text{-}2)_\mu$, (Beltran Martinez and Wittig 2024) $(a_\mu^{\rm HVP}(\text{NLO}) = -101.0(2.5)\times10^{-11})$



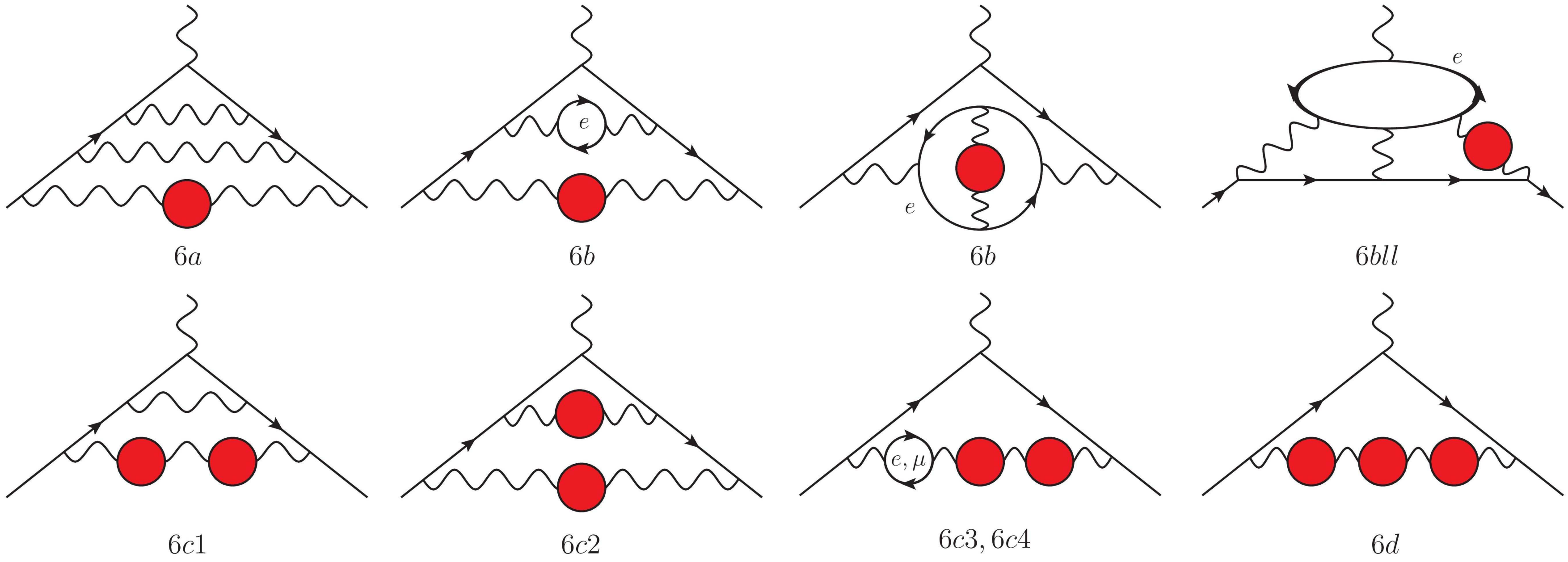
sample NNLO diagrams

- set 6a contains also diagrams with muon loops (like 6b 6bll)

- HVP with internal corrections already incorporated in NLO and LO



- $a_\mu^{\rm HVP}(\text{NNLO}; total) = +12.4(1) \times 10^{-11}$ Kurz Liu Marquard Steinhauser 2014



$$a_\mu^{\rm HVP}({\rm NNLO}; 6a) = \frac{\alpha^3}{\pi^4} \int_{m_\mu^2}^{\infty} \frac{ds}{s} \, K^{(6)}(s/m_\mu^2) \, {\rm Im}\Pi(s) = -\frac{\alpha^3}{\pi^4} \int_{-\infty}^{0} \frac{dt}{t} \Pi(t) \, {\rm Im} K^{(6)}(t/m_\mu^2)$$

- NLO: $K^{(4)}(s/m_\mu^2)$ is known analytically
- NNLO: $K^{(6)}(s/m_\mu^2)$ is NOT known analytically.
- Only a few terms of the *asymptotic expansions* for large s are known.
- We need to find *approximated* spacelike kernels from the asymptotic expansions

$K^{(6a)}(s/m_\mu^2)$: Only the first 4 terms of the expansion in power series of $r = m_\mu^2/s$ are known →n=4

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The expansion in small r contain terms with $r^n \ln r$, $r^n \ln^2 r$ and $r^n \ln^3 r$. We use an integral ansatz:

$$K^{(6a)}(s/m_\mu^2) = r \int_0^1 d\xi \left[\frac{L^{(6a)}(\xi)}{\xi + r} + \frac{P^{(6a)}(\xi)}{1 + r\xi} \right] \qquad L^{(6a)}(\xi) = G^{(6a)}(\xi) + H^{(6a)}(\xi) \ln \xi + J^{(6a)}(\xi) \ln^2 \xi \quad \text{new@NNLO}$$

$G^{(6a)}$, $H^{(6a)}$,$J^{(6a)}$, $P^{(6a)}$ polynomials of degree 3

$$G^{(6a)}(\xi) = \sum_{i=0}^{3} g_i^{(6a)} \xi^i, \quad H^{(6a)}(\xi) = \sum_{i=0}^{3} h_i^{(6a)} \xi^i, \quad J^{(6a)}(\xi) = \sum_{i=0}^{3} j_i^{(6a)} \xi^i, \quad P^{(6a)}(\xi) = \sum_{i=0}^{3} p_i^{(6a)} \xi^i$$

We integrate in ξ, expand in r, and we fit the coefficients $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$ and $p_i^{(6a)}$, $i = 0, 1, 2, 3$, in order to match the coefficients of the asymptotic expansion in r of $K^{(6a)}(s/m_\mu^2)$. The approximated kernel $\bar{\kappa}^{(6a)}(x)$ is

$$a_\mu^{HVP}(\mathrm{NNLO}; 6a) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 dx\, \bar{\kappa}^{(6a)}(x)\, \Delta\alpha_{\mathrm{had}}(t(x)),$$

$$\bar{\kappa}^{(6a)}(x) = \begin{cases} \frac{2-x}{x(1-x)}\, P^{(6a)}\!\left(\frac{x^2}{1-x}\right), & 0 < x < x_\mu = (\sqrt{5}-1)/2 = 0.618\ldots \\ \frac{2-x}{x^3}\, L^{(6a)}\!\left(\frac{1-x}{x^2}\right), & x_\mu < x < 1 \quad \text{discontinuous in } x_\mu \end{cases}$$

- The contributions of classes $(6b)$ and $(6bll)$ can be calculated similarly to class $(6a)$.
- $a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; 6a) = +8.0 \times 10^{-11}$ $\quad a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; 6b) = -4.1 \times 10^{-11}$ $\quad a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; 6bll) = +9.1 \times 10^{-11}$
- The uncertainty due to the series approximations of $K^{(6a)}$, $K^{(6b)}$, $K^{(6bll)}$ are estimated to be less than $O(10^{-12})$

(6a)

$j_0 = 0;$	$h_0 = -\frac{359}{36};$
$j_1 = -\frac{3793}{864};$	$h_1 = \frac{122293}{5184};$
$j_2 = \frac{35087}{21600};$	$h_2 = -\frac{43879427}{648000};$
$j_3 = \frac{1592093}{43200};$	$h_3 = \frac{14388407}{48000};$

$$g_0 = \frac{1301}{144} - \frac{19\pi^2}{9};$$
$$g_1 = \frac{441277}{10368} + \pi^2\left(-\frac{355}{648} + \ln 4\right) + \frac{25\ \zeta(3)}{2};$$
$$g_2 = -\frac{5051645167}{38880000} + \pi^2\left(\frac{221411}{32400} - 18\ln 2\right) - \frac{3919\ \zeta(3)}{60};$$
$$g_3 = \frac{14588342017}{38880000} + \pi^2\left(-\frac{2479681}{64800} + 112\ln 2\right) + \frac{3113\ \zeta(3)}{10};$$

$$p_0 = -\frac{1808080780513}{14580000} + \frac{41851\pi^4}{15} + \frac{8432\ln^4 2}{3} + 67456\ a_4 + \frac{2085448\ \zeta(3)}{15} + \\ +\pi^2\left(-\frac{11944163099}{194400} + \frac{272}{3}(180 - 31\ln 2)\ln 2 + \frac{115072\ \zeta(3)}{3}\right) - \frac{575360\ \zeta(5)}{3};$$
$$p_1 = \frac{134017456919}{96000} - \frac{4481182\pi^4}{135} - \frac{98420\ln^4 2}{3} - 787360\ a_4 + 2255200\ \zeta(5) + \\ +\pi^2\left(\frac{23549054249}{32400} - 201122\ln 2 + \frac{98420\ln^2 2}{3} - 451040\ \zeta(3)\right) - \frac{57189259\ \zeta(3)}{36};$$
$$p_2 = -\frac{13069081405453}{3888000} + \frac{330073\pi^4}{4} + 80790\ln^4 2 + 1938960\ a_4 + \frac{77371609\ \zeta(3)}{20} + \\ +\pi^2\left(-\frac{729995599}{405} + 6(85313 - 13465\ln 2)\ln 2 + 1114360\ \zeta(3)\right) - 5571800\ \zeta(5);$$
$$p_3 = \frac{1274611832039}{583200} - \frac{986377\pi^4}{18} - 53340\ln^4 2 - 1280160\ a_4 + \frac{11057200\ \zeta(5)}{3} + \\ +\pi^2\left(\frac{5809659289}{4860} + 420\ln 2(-823 + 127\ln 2) - \frac{2211440\ \zeta(3)}{3}\right) - \frac{22833188\ \zeta(3)}{9};$$

Table 1: The coefficients $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$, $p_i^{(6a)}$ $(i = 0, 1, 2, 3)$. The superscript $(6a)$ has been dropped for simplicity. In the above coefficients, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$ and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.

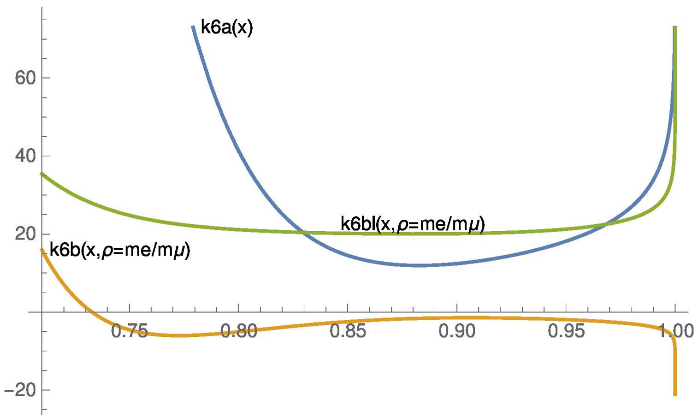
(6b)

$j_0 = 0;$	$h_0 = \frac{65}{54};$
$j_1 = \frac{11}{27};$	$h_1 = -\frac{3559}{1296} + \rho^2 + \frac{5}{18}\ln\rho;$
$j_2 = \frac{41}{120};$	$h_2 = \frac{3917}{432} - \frac{82\rho^2}{3} + \frac{61}{10}\ln\rho;$
$j_3 = -\frac{507}{40};$	$h_3 = -\frac{4109}{80} + \frac{2211\rho^2}{10} - \frac{1763}{30}\ln\rho;$

$$g_0 = \frac{1}{108}\left(259 - 72\rho^2 + 276\ln\rho\right);$$
$$g_1 = -\frac{9215}{1296} + \frac{65\pi^2}{162} - \frac{3\pi^2\rho}{4} + \frac{49\rho^2}{36} + \left(-\frac{301}{54} + 8\rho^2\right)\ln\rho + \frac{4}{3}\ln^2\rho + 2\ \zeta(3);$$
$$g_2 = \frac{501971}{40500} - \frac{113\pi^2}{36} + \frac{270\pi^2\rho}{36} - \frac{8417\rho^2}{180} + \left(\frac{3479}{900} - 44\rho^2\right)\ln\rho - 8\ln^2\rho - 12\ \zeta(3);$$
$$g_3 = -\frac{2523823}{324000} + \frac{625\pi^2}{36} - 49\pi^2\rho + \frac{84946\rho^2}{225} + \left(\frac{987}{50} + 200\rho^2\right)\ln\rho + \frac{112}{3}\ln^2\rho + 56\ \zeta(3);$$

$$p_0 = -\frac{95519053063}{486000} - 7275\pi^2\rho + \left(-\frac{587150693}{5400} + \frac{75272\rho^2}{3} + \frac{120800\pi^2}{9}\right)\ln\rho + \left(\frac{1135508}{9} + 96\rho^2\right)\zeta(3) + \\ +4720\ln^2\rho + \frac{1067115409\rho^2}{5400} + \pi^2\left(\frac{24382331}{810} - \frac{285184}{9}\ln 2\right) - 32\pi^2\rho^2(687 + \ln 4);$$
$$p_1 = \frac{279489728279}{121500} + \frac{179283\pi^2\rho}{2} + \left(\frac{2280933773}{1800} - 309540\rho^2 - \frac{1419328\pi^2}{9}\right)\ln\rho - \frac{10}{3}(446023 + 216\rho^2)\zeta(3) + \\ -\frac{174712}{3}\ln^2\rho - \frac{174350167\rho^2}{75} + \pi^2\left(-\frac{143574463}{405} + \frac{3352256\ln 2}{9}\right) + \frac{16}{3}\pi^2\rho^2(48481 + 90\ln 2);$$
$$p_2 = -\frac{229560199193}{40500} - \frac{912495\pi^2\rho}{4} + \left(-\frac{1867939691}{600} + 788488\rho^2 + \frac{1168336\pi^2}{3}\right)\ln\rho + \left(\frac{11034553}{3} + 1440\rho^2\right)\zeta(3) + \\ +148348\ln^2\rho + \frac{258653648\rho^2}{45} + \frac{4}{135}\pi^2(29597029 - 31048560\ln 2) - \frac{320}{3}\pi^2\rho^2(5989 + \ln 512);$$
$$p_3 = \frac{72762177677}{19440} + 154035\pi^2\rho - \frac{7}{108}(-31650719 + 3973440\pi^2 + 8220240\rho^2)\ln\rho - \frac{280}{9}(78283 + 27\rho^2)\zeta(3) + \\ -100240\ln^2\rho - \frac{513692207\rho^2}{135} + \frac{35}{162}\pi^2(-2687659 + 2816064\ln 2) + \frac{140}{3}\pi^2\rho^2(9055 + \ln 4096);$$

Table 2: The coefficients $g_i^{(6b)}$, $h_i^{(6b)}$, $j_i^{(6b)}$, $p_i^{(6b)}$ $(i = 0, 1, 2, 3)$. The superscript $(6b)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.



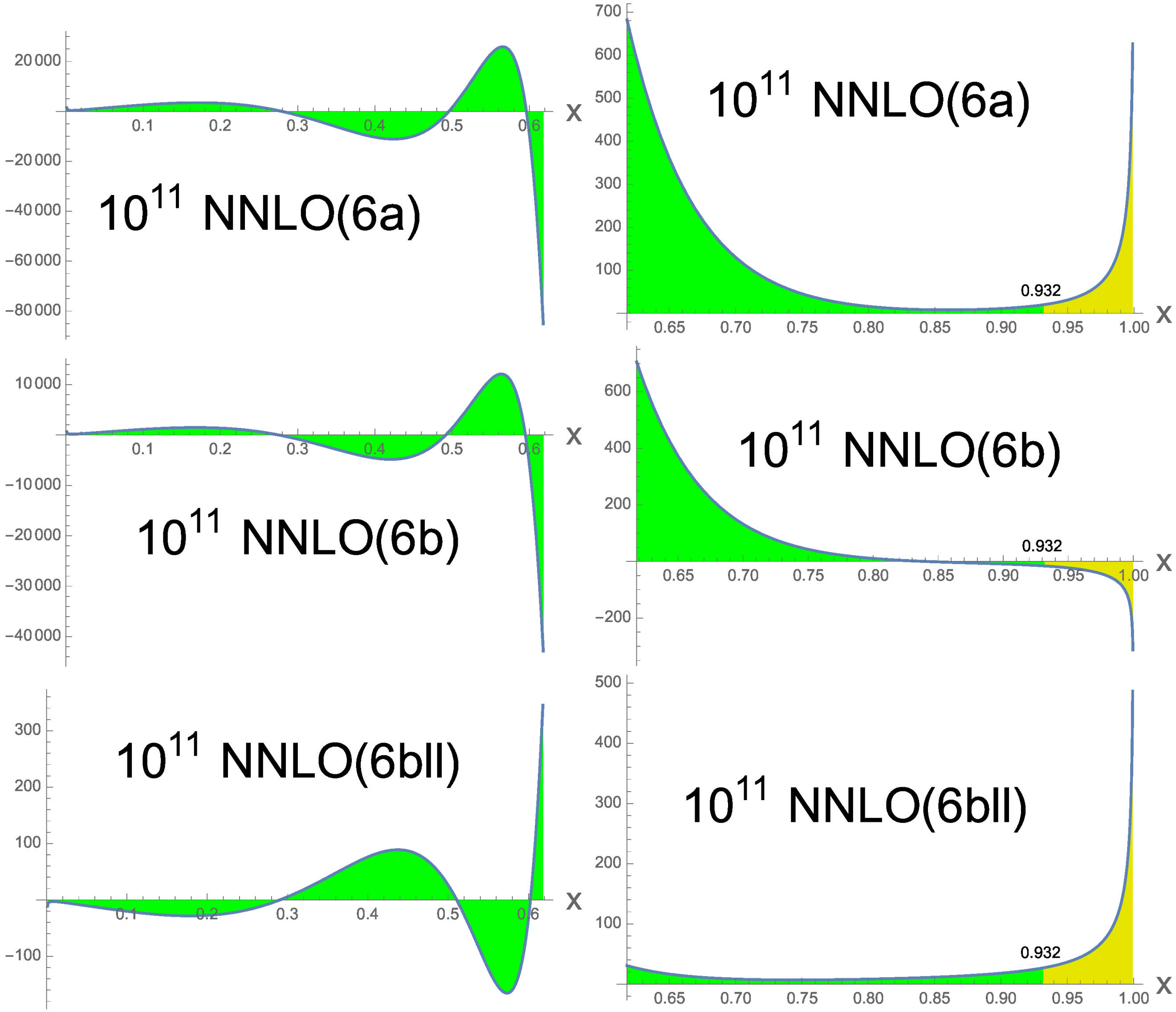
(6bll)

$j_0 = 0;$	$h_0 = -\frac{9}{2};$
$j_1 = \frac{4}{27} - \frac{9\rho^2}{2};$	$h_1 = \frac{59}{9} - \frac{275\rho^2}{36} - 18\rho^2\ln\rho;$
$j_2 = -\frac{41}{48} + \frac{2201\rho^2}{216};$	$h_2 = -\frac{485}{32} + \frac{1351\rho^2}{48} + \frac{659\rho^2}{18}\ln\rho;$
$j_3 = \frac{3037}{900} - \frac{5909\rho^2}{216};$	$h_3 = \frac{282617}{6750} - \frac{10481\rho^2}{108} - \frac{851\rho^2}{9}\ln\rho;$

$$g_0 = \frac{43}{8} - 4\pi^2\rho + 15\rho^2 + \pi^2\rho^2 - 18\rho^2\ln\rho + 6\rho^2\ln^2\rho;$$
$$g_1 = -\frac{73}{81} + \frac{8\pi^2}{81} + \frac{40\pi^2\rho}{9} + \frac{2437\rho^2}{108} + \frac{17\pi^2\rho^2}{9} + \frac{607\rho^2}{18}\ln\rho - \frac{20\rho^2}{3}\ln^2\rho + \frac{2}{3}\zeta(3) + 2\rho^2\zeta(3);$$
$$g_2 = -\frac{385}{162} - \frac{41\pi^2}{72} - \frac{28\pi^2\rho}{3} - \frac{89873\rho^2}{5184} - \frac{997\pi^2\rho^2}{324} - \frac{1961\rho^2}{72}\ln\rho + 14\rho^2\ln^2\rho - \frac{5}{2}\zeta(3) - \frac{16\rho^2}{3}\zeta(3);$$
$$g_3 = \frac{2691761}{202500} + \frac{3037\pi^2}{1350} + 24\pi^2\rho + \frac{655429\rho^2}{97200} + \frac{2359\pi^2\rho^2}{324} + \frac{6943\rho^2}{360}\ln\rho - 36\rho^2\ln^2\rho + \frac{42}{5}\zeta(3) + 15\rho^2\zeta(3);$$

$$p_0 = -\frac{343277101}{45000} - \frac{33156604927\rho^2}{583200} + \pi^2\left(-\frac{615427}{4050} + \frac{6776\rho}{3} + \frac{763121\rho^2}{972}\right) - \frac{4\pi^4}{135}(7817 + 3212\rho^2) + \\ +\left(-\frac{7290521}{3240} + \frac{49622\pi^2}{27} - \frac{128\pi^4}{9}\right)\rho^2\ln\rho + \left(-3388 - \frac{80\pi^2}{3}\right)\rho^2\ln^2\rho + \\ +\left(25642 + \frac{1515724\rho^2}{27} - 128\pi^2\rho^2 - 160\rho^2\ln\rho\right)\zeta(3) - \frac{1280}{3}\rho^2\zeta(5);$$
$$p_1 = \frac{89280434843}{972000} + \frac{248834878697\rho^2}{388800} - \frac{1}{324}\pi^2(-533001 + 9110736\rho + 3110417\rho^2) + \frac{2}{135}\pi^4(180247 + 73530\rho^2) + \\ +\left(\frac{11101973}{1080} - \frac{193400\pi^2}{9} + \frac{320\pi^4}{3}\right)\rho^2\ln\rho + \frac{2}{3}(63269 + 300\pi^2)\rho^2\ln^2\rho + \\ +\frac{1}{45}\left(-13410977 + 100(-292301 + 432\pi^2)\rho^2 + 54000\rho^2\ln\rho\right)\zeta(3) + 3200\rho^2\zeta(5);$$
$$p_2 = -\frac{6209532853}{27000} - \frac{29997466847\rho^2}{19440} + \pi^2\left(-\frac{114521}{30} + 71840\rho + \frac{1970140\rho^2}{81}\right) - \frac{4}{9}\pi^4(14685 + 6032\rho^2) + \\ -\frac{1}{54}(190613 - 2847360\pi^2 + 11520\pi^4)\rho^2\ln\rho - 80(1347 + 5\pi^2)\rho^2\ln^2\rho + \\ -\frac{10}{9}\left(-658509 + (-1431463 + 1728\pi^2)\rho^2 + 2160\rho^2\ln\rho\right)\zeta(3) - 6400\rho^2\zeta(5);$$
$$p_3 = \frac{497263311179}{324000} + \frac{7324831423\rho^2}{7290} + \pi^2\left(\frac{3897971}{1620} - \frac{145880\rho}{3} - \frac{3977785\rho^2}{243}\right) + \frac{14}{27}\pi^4(8269 + 3419\rho^2) + \\ +\frac{7}{81}(-81551 - 401520\pi^2 + 1440\pi^4)\rho^2\ln\rho + \frac{140}{3}(1563 + 5\pi^2)\rho^2\ln^2\rho + \\ +\frac{35}{27}\left(-371889 + 16(-50437 + 54\pi^2)\rho^2 + 1080\rho^2\ln\rho\right)\zeta(3) + \frac{11200}{3}\rho^2\zeta(5);$$

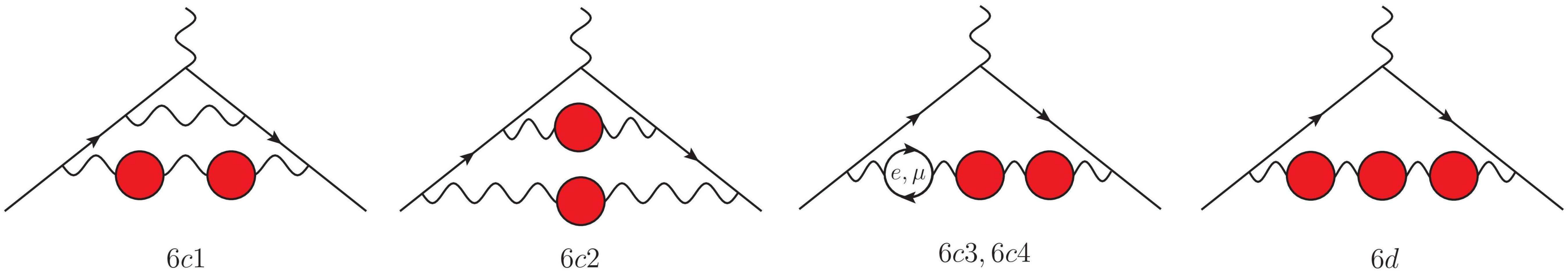
Table 3: The coefficients $g_i^{(6bll)}$, $h_i^{(6bll)}$, $j_i^{(6bll)}$, $p_i^{(6bll)}$ $(i = 0, 1, 2, 3)$. The superscript $(6bll)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.



Plot of spacelike NNLO integrands $6a$ $6b$ $6bll$

Huge, almost complete cancellations between positive and negative parts of integrands

Part of the integral directly scanned by MUonE: $6a$: 15%, $6b$: 16%, $6bll$: 38%.



$$a_\mu^{HVP}(\text{NNLO}; 6c) = a_\mu^{HVP}(\text{NNLO}; 6c1) + a_\mu^{HVP}(\text{NNLO}; 6c2) + a_\mu^{HVP}(\text{NNLO}; 6c3) + a_\mu^{HVP}(\text{NNLO}; 6c4)$$

$$a_\mu^{HVP}(\text{NNLO}; 6c1) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \mathrm{d}x \left[\kappa^{(4)}(x) - \frac{2\pi}{\alpha}\kappa^{(2)}(x)\, \Delta\alpha_\mu^{(2)}(t(x))\right] \left[\Delta\alpha_{\text{had}}(t(x))\right]^2$$

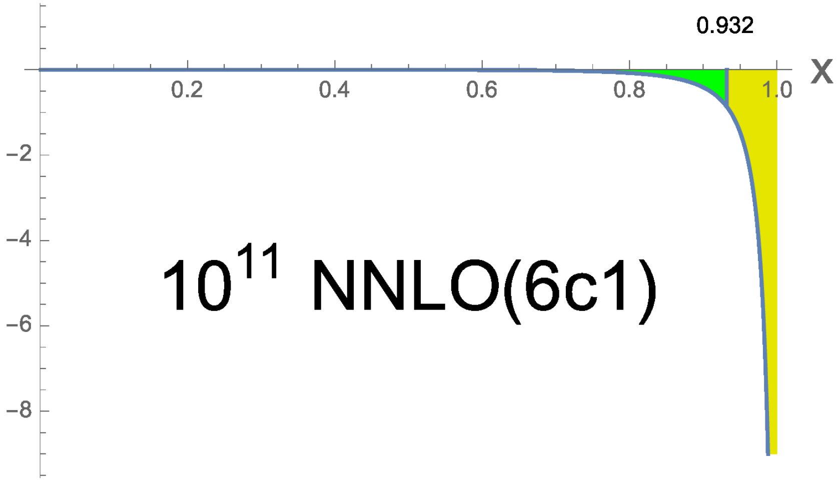
6c4 separated
multiplicity=3

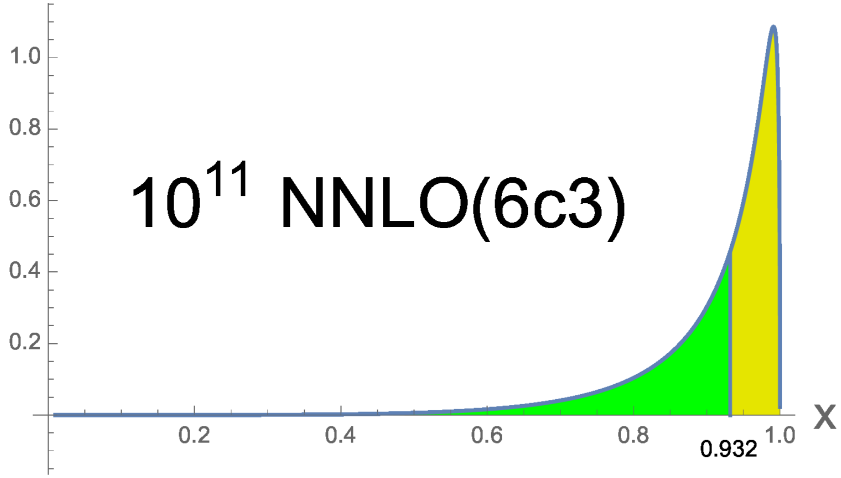
$$a_\mu^{HVP}(\text{NNLO}; 6c3) = \frac{3\alpha}{\pi} \int_0^1 \mathrm{d}x\, \kappa^{(2)}(x) \left[\Delta\alpha_{\text{had}}(t(x))\right]^2 \Delta\alpha_e^{(2)}(t(x))$$

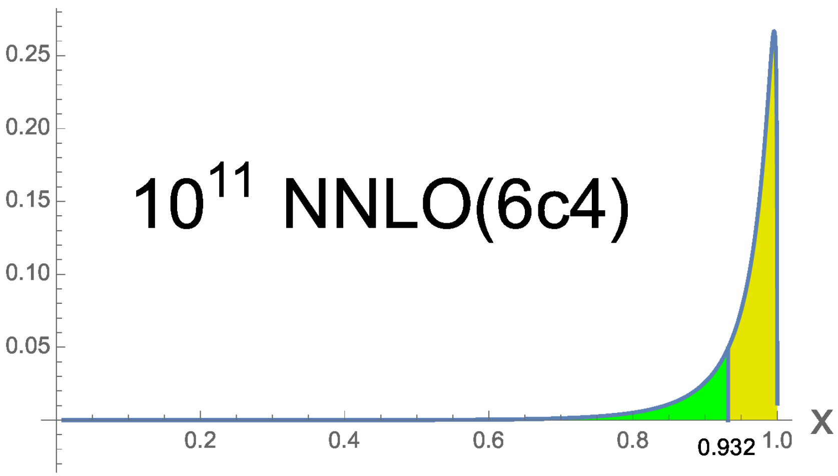
$$a_\mu^{HVP}(\text{NNLO}; 6c4) = \frac{3\alpha}{\pi} \int_0^1 \mathrm{d}x\, \kappa^{(2)}(x) \left[\Delta\alpha_{\text{had}}(t(x))\right]^2 \Delta\alpha_\mu^{(2)}(t(x))$$

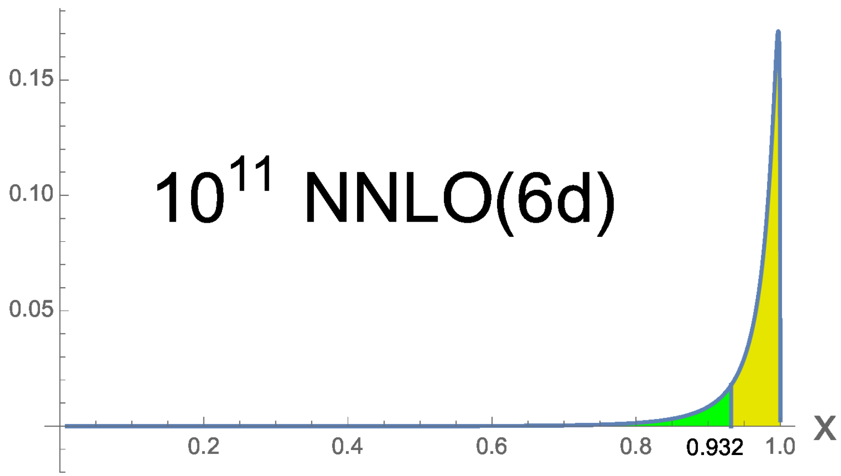
$$a_\mu^{HVP}(\text{NNLO}; 6d) = \frac{\alpha}{\pi} \int_0^1 \mathrm{d}x\, \kappa^{(2)}(x) \left[\Delta\alpha_{\text{had}}(t(x))\right]^3$$

$a_\mu^{HVP}(6c1) = -5 \times 10^{-12}$, $a_\mu^{HVP}(6c3) = 0.9 \times 10^{-12}$, $a_\mu^{HVP}(6c4) = 0.1 \times 10^{-12}$, $a_\mu^{HVP}(6d) = 0.05 \times 10^{-12}$



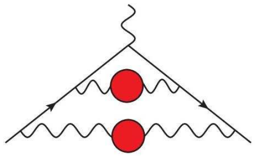






peak at $x = 0.997$

Part of the integral directly scanned by MUonE: $6c1 : 9\%$, $6c3 : 44\%$, $6c4 : 23\%$, $6d : 16\%$. 6c2 ?



This class requires *double* integrals

$$a_\mu^{HVP}(\mathrm{NNLO};6c2) = \frac{\alpha^2}{\pi^4}\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s}\int_{s_0}^{\infty}\frac{\mathrm{d}s'}{s'}K^{(6c2)}(s/m_\mu^2, s'/m_\mu^2)\mathrm{Im}\Pi_{\mathrm{had}}(s)\,\mathrm{Im}\Pi_{\mathrm{had}}(s').$$

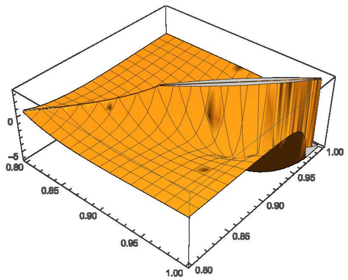
$$a_\mu^{HVP}(\mathrm{NNLO};6c2) = \left(\frac{\alpha}{\pi}\right)^2\int_{x_\mu}^{1}\mathrm{d}x\int_{x_\mu}^{1}\mathrm{d}x'\,\bar\kappa^{(6c2)}(x,x')\Delta\alpha_{\mathrm{had}}(t(x))\Delta\alpha_{\mathrm{had}}(t(x')),$$

$\bar\kappa^{(6c2)}(x,x')$ space-like bidimensional kernel, $x_\mu < \{x, x'\} < 1$

$$\bar\kappa^{(6c2)}(x,x') = \frac{2-x}{x^3}\frac{2-x'}{x'^3}G^{(6c2)}\left(\frac{1-x}{x^2},\frac{1-x'}{x'^2}\right)$$

From the leading terms of the known asymptotic expansion of $K^{(6c2)}(s/m_\mu^2, s'/m_\mu^2)$:
$s/s' \ll 1$ or $s/s' \approx 1$ or $s/s' \gg> 1$ and $s, s' \gg m_\mu^2$ we get the approximated space-like kernel

$$G^{(6c2)}(\xi,\xi') = \frac{1855-188\pi^2}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')}{\max(\xi,\xi')^2} + \frac{988\pi^2-9765}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')^2}{\max(\xi,\xi')^3} + \frac{6\,(435-44\pi^2)}{32\pi^2-315}\frac{\min(\xi,\xi')^3}{\max(\xi,\xi')^4}$$



plot of $\bar\kappa^{(6c2)}(x,x')$

Contribution of 6c2 class is $a_\mu^{HVP}(6c2) = -1.8\times10^{-12}$
The uncertainty of this leading order approximation is estimated to be $\sim 10^{-13}$
NNLO(6c2): part of the integral directly scanned by MUonE= 6% of the diagram contribution

- NLO: Exact NLO space-like kernels are known.
- MUonE directly scans 41%, 82%, and 49% of the integrals of NLO $(4a)$, $(4b)$ and $(4c)$, respectively
- Using the alternative approach on the timelike integral the percentages of the contributions deduced from the MUonE data can be substantially improved *(work in progress)*.
- For the NLO(4a) time-kernel we have an analytical expansion in powers of T and a numerical expansion in powers of $(\hat{T}_0/\hat{T} - 1)$
- The combination of these expansions, with a suitable choice of numbers of terms, of the expansion point $\hat{T}_0$ and of the separation point $\hat{T}_s$ between regimes, allows to determine the NLO(4a) time-kernel with an error $\Delta\tilde{f} < 3 \times 10^{-8}$ for every value of $\hat{T}$. These expansions were already used in a lattice determination of the NLO HVP contribution to muon g-2.
- NNLO: Approximated space-like NNLO kernels were obtained from the first terms of the asymptotic expansions. For one set (6c2) containing two HVP insertions on *different* photon lines, we worked out a *bidimensional* approximated space-like kernel. The precision of the contributions of all the approximated space-like kernels obtained is at the level of 10^{-13}.

Thank You