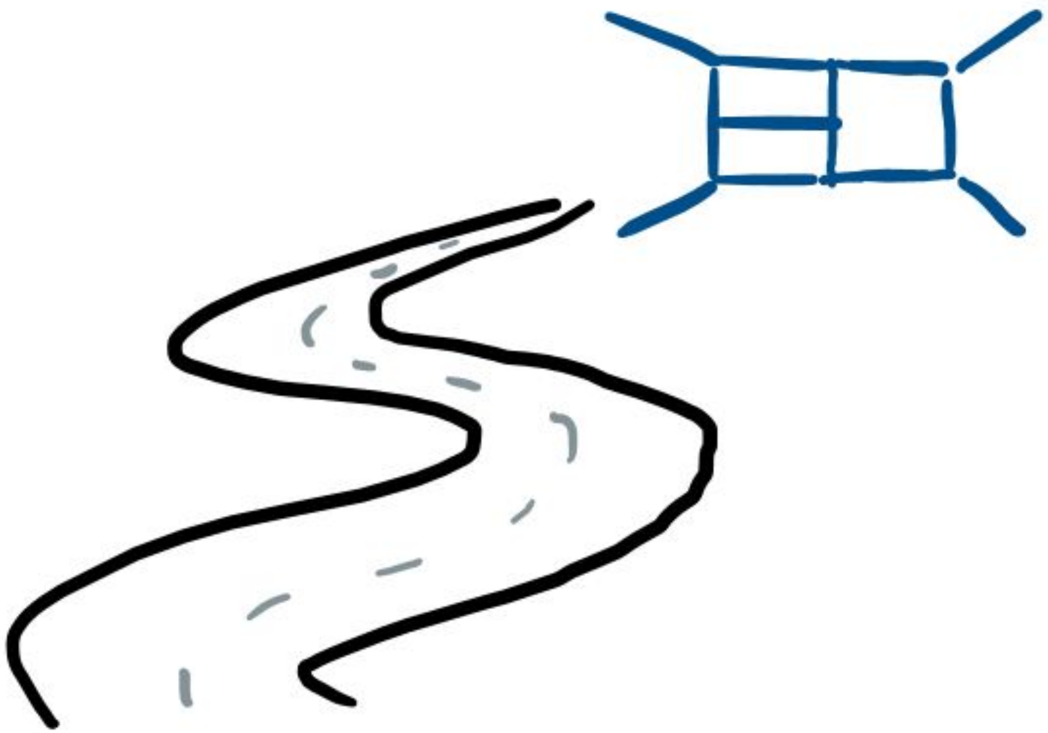
The road to N^3LO



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- Tensor Decomposition
- Algorithm for grouping Feynman diagrams via loop momenta shifts
- Results and future plans

See Marco Bonetti's talk from MPP2024 for related insights.

Motivation

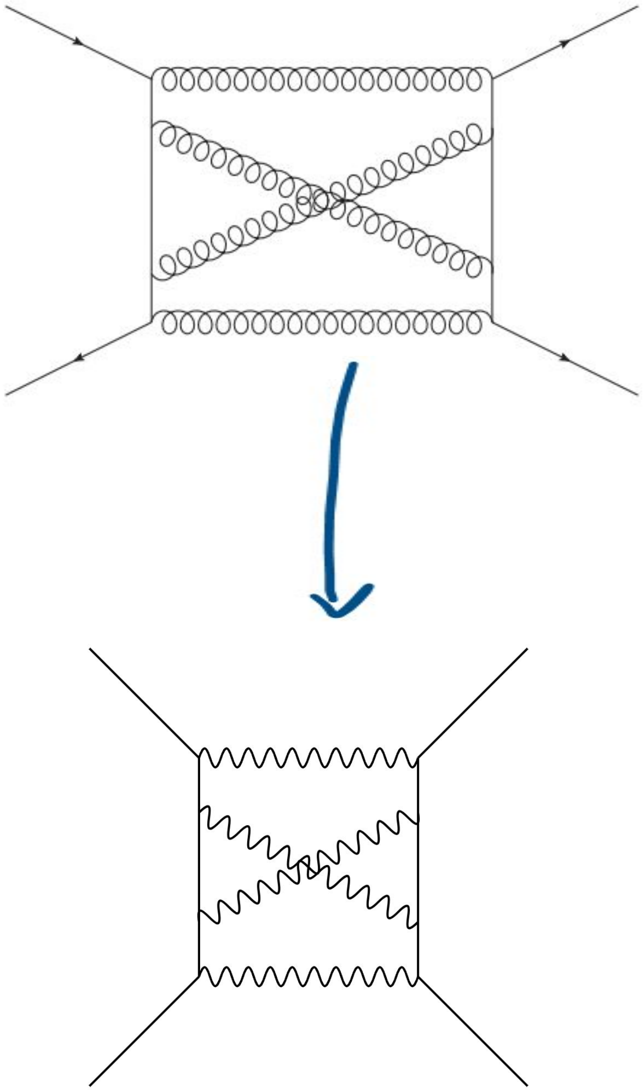
Calculations for massless four-fermion and two-quark two-gluon scatterings in QCD have been completed up to the finite remainder.

Similar calculations in a QED framework have not yet been completed.

Problem!

It is not easy due to the number of diagrams and the complexity of the expressions involved. This causes computational issues both with time and memory.

Can we take results from existing QCD calculations?



In QCD, similar calculations have already been performed.

As QED is Abelian, can we just abelianise these expressions?

$$e^+e^- \to \gamma\gamma$$

Not as straightforward as it might seem...

$$-2 \times \Big(\langle \mathcal{M}^{(2)}_{gg} | \mathcal{C}_1 \rangle_{[-1,1]} + \langle \mathcal{M}^{(2)}_{gg} | \mathcal{C}_2 \rangle_{[-1,1]} + 2 \times \langle \mathcal{M}^{(2)}_{gg} | \mathcal{C}_3 \rangle_{[-2,1]} \Big) \; .$$

How can we make the calculation easier?

Fortunately, there are many modern techniques that we can now use:

- Tensor Decomposition
- Grouping Diagrams at integrand level
- Integration-by-parts relations

And many more...

Tensor Decomposition

Tensor Decomposition is the process of breaking down a scattering amplitude into the Lorentz structures that describe the relevant process.

$\mathcal{A}_B^{(l)}$ → $\mathcal{T}_i^{(l)}$

4-dimensional external states

$\mathcal{A}_B^{(l)}$ ↓ $\mathcal{F}_{B;i}^{(l)}$

D-dimensional internal states

Using Tensor decomposition we obtain D-dimensional form factors comprised of scalar products.
We can then perform all intermediary steps on these expressions. Before recovering the dependence on external states in the final stages of the calculation.

Not only useful for Helicity Amplitudes, but also interference at loop level.

[Chen (2019), Peraro, Tancredi (2020)]

Tensor Decomposition

We recover expressions for the amplitude by summing the products of the form factors with their corresponding tensor structures

$$\mathcal{A}^{(l)} = \sum_{i=1}^{n} \mathcal{F}_i^{(l)} \, \mathcal{T}_i^{(l)}$$

By considering the external states to be in 4-dimensions we also reduce the number of independent tensor structures.

	D-Dimensional space	4-Dimensional space
$\bar{u}(p_2)\gamma_{\mu_1} u(p_1)\, \bar{u}(p_4)\gamma^{\mu_1} u(p_3)$ $\bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} u(p_1)\, \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} u(p_3)$	✗ (Dependent)	✓ (Independent)

Helicity Amplitudes

Helicity amplitudes are a compact way of representing scattering amplitudes in specific helicity states. Reducing the complexity of summing over spin states.

$e^+e^- \to \mu^+\mu^-$	$\mathcal{H}^{(l)}_{\mu\mu,(-++-)} = -2\mathcal{F}^{(l)}_{\mu\mu,1} + (1-x)\,\mathcal{F}^{(l)}_{\mu\mu,2}\,,$ $\mathcal{H}^{(l)}_{\mu\mu,(-+-+)} = -2\mathcal{F}^{(l)}_{\mu\mu,1} - x\mathcal{F}^{(l)}_{\mu\mu,2}\,.$
$e^+e^- \to \gamma\gamma$	$\mathcal{H}^{(l)}_{\gamma\gamma,(-+--)} = -\frac{1}{x}\mathcal{F}^{(l)}_{\gamma\gamma,2} - \frac{(1-x)}{2x}\mathcal{F}^{(l)}_{\gamma\gamma,3} - \frac{1}{x}\mathcal{F}^{(l)}_{\gamma\gamma,4}\,,$ $\mathcal{H}^{(l)}_{\gamma\gamma,(-+++)} = \frac{1}{x}\mathcal{F}^{(l)}_{\gamma\gamma,1} - \frac{(1-x)}{2x}\mathcal{F}^{(l)}_{\gamma\gamma,3} - \frac{1}{x}\mathcal{F}^{(l)}_{\gamma\gamma,4}\,,$ $\mathcal{H}^{(l)}_{\gamma\gamma,(-+-+)} = \frac{(1-x)}{2x}\mathcal{F}^{(l)}_{\gamma\gamma,3} + \frac{(1-x)}{x}\mathcal{F}^{(l)}_{\gamma\gamma,4}\,,$ $\mathcal{H}^{(l)}_{\gamma\gamma,(-++-)} = -\frac{1}{x}\mathcal{F}^{(l)}_{\gamma\gamma,1} + \frac{1}{x}\mathcal{F}^{(l)}_{\gamma\gamma,2} + \frac{(1-x)}{2x}\mathcal{F}^{(l)}_{\gamma\gamma,3} + \frac{(1-x)}{x}\mathcal{F}^{(l)}_{\gamma\gamma,4}\,.$

Algorithm for grouping diagrams at integrand level

Integration-by-parts relations are extremely beneficial but can provide a bottleneck in calculations.

$$\int \frac{\partial}{\partial q^\alpha}\left(k^\alpha \prod \frac{1}{D_j^{a_j}}\right) dk = 0, \quad k^\alpha = q^\alpha, p^\alpha$$

[Laporta 2001]

Grouping diagrams that obey the same IBPs at integrand level means we can reduce the number of IBPs we need to generate.

3L Examples	Before grouping	After grouping
$e^+e^- \to \mu^+\mu^-$	1008 diagrams	158 families
$e^+e^- \to \gamma\gamma$	2268 diagrams	176 families

Results and future plans

Successfully found analytical expressions for massless QED processes at NNLO up to ϵ^2.

$$e^+e^- \to \mu^+\mu^- \,,$$

$$e^+\mu^- \to e^+\mu^- \,,$$

$$e^+e^- \to \mu^+\mu^- \,,$$

$$e^+e^- \to \gamma\gamma \,,$$

This is an important step towards the 3L calculation!

Helicity amplitudes in massless QED to higher orders in the dimensional regulator

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We analytically calculate one- and two-loop helicity amplitudes in massless QED, by adopting a four-dimensional tensor decomposition. We draw our attention to four-fermion and Compton scattering processes to higher orders in the dimensional regulator, as required for theoretical predictions at N^{3}LO. We organise loop amplitudes by proposing an efficient algorithm at integrand level to group Feynman graphs into integral families. We study the singular structure of these amplitudes and discuss the correspondence between QED and QCD processes. We present our results in terms of generalised polylogarithms up to transcendental weight six.

(2411.07063)

Diagram	**PL**	**NPL**
D_1	$(q_1)^2$	$(q_1)^2$
D_2	$(q_1 - p_2 - p_3 - p_4)^2$	$(q_1 - p_2 - p_3 - p_4)^2$
D_3	$(q_1 - p_3 - p_4)^2$	$(p_4 - q_1 + q_2)^2$
D_4	$(q_2)^2$	$(q_2)^2$
D_5	$(q_2 - p_3 - p_4)^2$	$(q_2 - p_2 - p_3)^2$
D_6	$(q_2 - p_4)^2$	$(q_2 - p_3)^2$
D_7	$(q_1 - q_2)^2$	$(-q_1 + q_2)^2$
D_8	$(q_1 - p_4)^2$	$(q_1 - p_3)^2$
D_9	$(q_2 - p_2 - p_3 - p_4)^2$	$(q_2 - p_2 - p_3 - p_4)^2$

TABLE II: Propagators for planar and non-planar diagrams in the configuration of external momenta of Fig. 1.

[TD and WJT. (2024)]

Results and future plans

Initial steps at N^3LO.

$$e^+e^- \to \mu^+\mu^-$$

We have managed to group all Feynman diagrams into integrand families via loop momenta shifts, and decompose them into form factors.

The next step is to perform IBP reduction into master integrals.

553 MIs!!

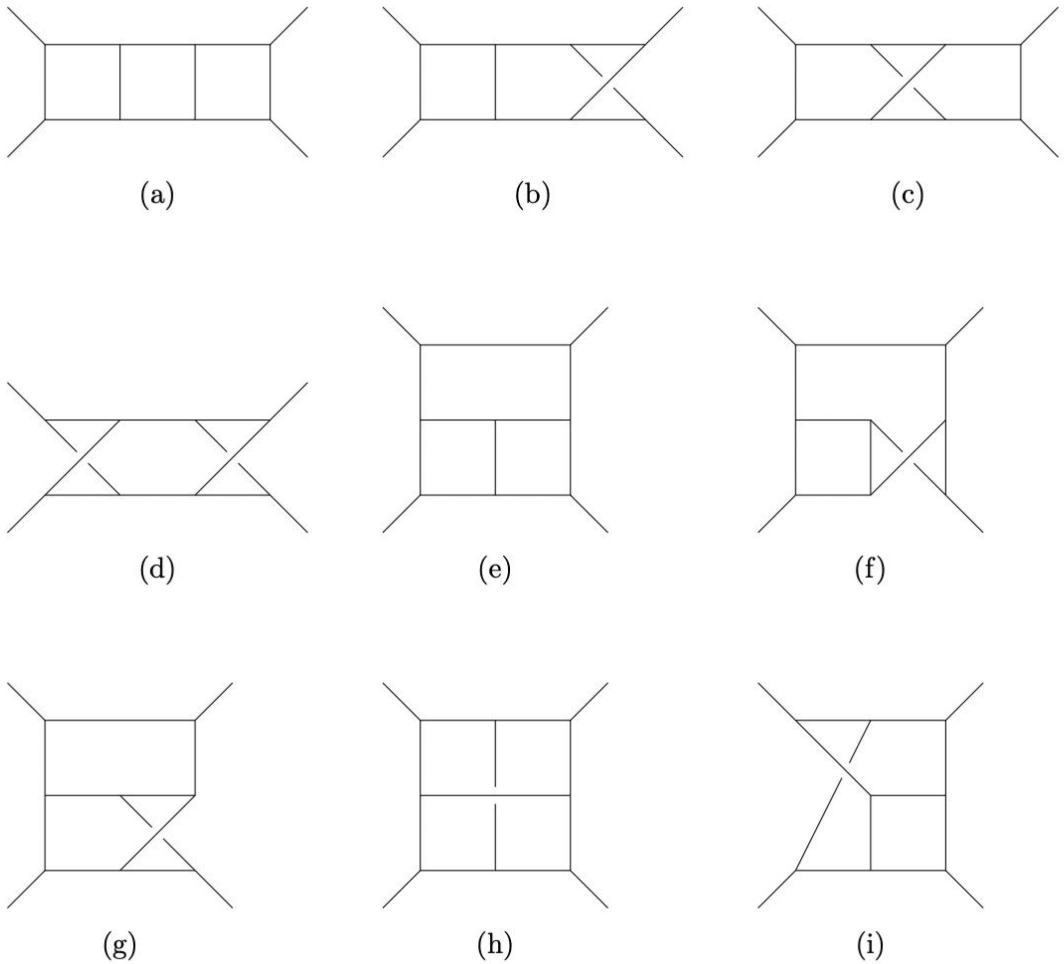

Figure 1. The nine integral families needed to describe all master integrals for three-loop massless four-particle scattering. The external legs are associated with the momenta p_1, p_3, p_4 and p_2 in clockwise order starting with the top left corner.

[Henn et al. (2020)]