Thinking about Natural Science

An introduction to philosophy for scientists
Lecture V

Paolo Beltrame
paolo.beltrame@liverpool.ac.uk

University of Liverpool
Department of Physics - Particle Physics Group

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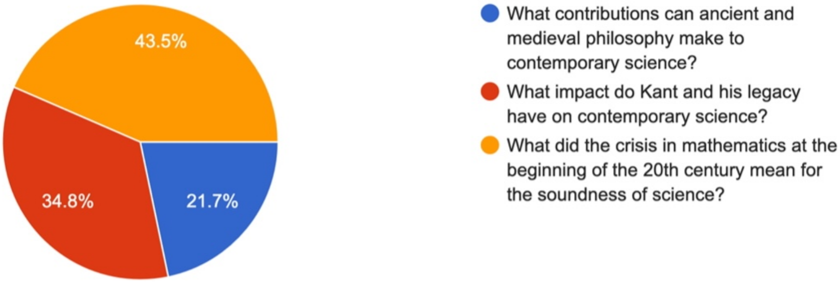
Overview of the course

I) [3 Feb. 2025] Introduction and quick historical background
II) [10 Feb. 2025] Modern science and philosophical difficulties...
III) [17 Feb. 2025] (Neo)Positivism, Popper and post-popperian debate
IV) [10 Mar. 2025] Case study: Laws of Physics, Reality, hints of the Truth
V) [24 Mar. 2025] Your preference
VI) [31 Mar. 2025] Guest lecture.

The choice for the V Lesson



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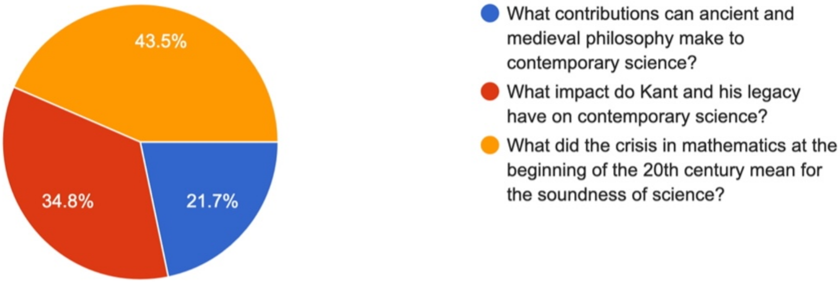

The p-value (for χ^2 test statistic $= 1.65$) is ~ 0.44 (much greater of $\alpha = 0.05$)
$\Rightarrow$ no strong statistical evidence to conclude that the 3^{rd} was the preferred choice

The crisis of the foundations of mathematics

Overview of today

1. Mathematics
2. Mathematical Platonism
3. Mathematical anti-Platonism
4. Logicism, intuitionism, and formalism
5. Kurt Gödel

Mathematics from philosophical perspective

Mathematics

Early Greek philosophy *axiomatic approach* $\rightarrow$ dispute as to which is more basic, arithmetic or geometry

- Pythagorean: only natural and rational numbers; then accepted $\sqrt{2}$
- Axiomatic method + notion of proof: Plato and Euclid's *Elements* (∼300 BCE)
  Foundations: a finite number of basic mathematical truths (*axioms* or *postulates*)
  Other true statements can be derived in a finite number of steps (*theorems*)

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  Logical and coherent reasoning $\rightarrow$ framework of *hypothetical-deductive* systems

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Logical and coherent reasoning $\rightarrow$ framework of *hypothetical-deductive* systems

Definitions of and *Axioms*

properties of defined objects
(*abstraction*)

$\Downarrow$

New certainties

Thinkers' view

Mathematics model for almost all other disciplines

$\Rightarrow$ R. Descartes (1596 - 1650), B. Spinoza (1632 - 1677), A. Comte (1798 - 1857)...

Galileo Galilei (1564 - 1642)
$\rightarrow$ *[The world] is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word*

Il Saggiatore

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Il Saggiatore

A universal and objective language: mathematics tells us how reality IS

The queen of the sciences, model for reasoning

Sartorius von Waltershausen

Immanuel Kant (1724 - 1804)

- Geometry and mathematics have insurmountable and unavoidable *limits*
- It is impossible to reach the *noumenon* (the 'dream' of every metaphysics)
- Mathematics and geometry, as well as all scientific knowledge, only reach the *phenomenon*
- However this knowledge is universal

$\Rightarrow$ *Mathematics is valid only in the reality the way we perceive it*

- It is our way of thinking and perceiving the world
- But it doesn't tell us anything about the noumenal dimension

Critique of Pure Reason (1781/1787)

Philosophy of mathematics

1. Meanings of ordinary mathematical sentences
   What is the best way to interpret standard mathematical concepts?
   (*3 is prime*, $2 + 2 = 4$ and *There are infinitely many prime numbers*)
   $\Rightarrow$ what certain expressions mean (or refer to) in ordinary discourse
   A semantic theory for the language of mathematics

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2. Existence of abstract objects
   How the reality of the mathematical concepts should be interpreted?
   Platonism: interpretation of sentences such as *4 is even*
   $\Rightarrow$ the sentence is true
   $\Rightarrow$ natural that abstract objects exist

Foundations of Mathematics

- Study of the logical and philosophical basis of mathematics
$\rightarrow$ Whether the axioms of a given system ensure its completeness and its consistency

Euclid's *Elements* $\rightarrow$ 2,000 years of mathematics perfectly solid
A set of formal logical arguments based on a few basic terms and axioms
$\rightarrow$ A systematic method of rational exploration well into the XIX century

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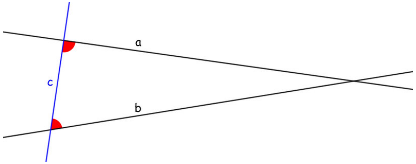
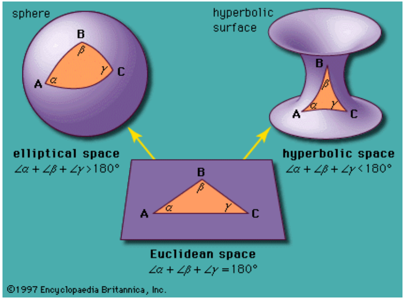
- Alternative non-Euclidean geometries $\Rightarrow$ Euclidean geometry not correspond with *reality*
- G. Cantor (1845 - 1918) and others $\Rightarrow$ set theory to answer paradoxical results and provide a more rigorous foundation for mathematics

Main problematic elements





Are all mathematical truths axioms or theorems? (Completeness)

Can it be determined 'mechanically' if a statement is a theorem? (Decidability)

Mathematical Platonism

Abstract objects

1. There exist abstract objects (non-spatio-temporal, non-physical, and non-mental) they have always existed and they always will exist

- We can have mental ideas of abstract objects (*e.g.* the number 4)
- But the number 4 is not just a mental idea
- They are unchanging and entirely noncausal
(not made of physical matter and not derived from the observation of physical objects)

G. Frege (1848 - 1925), K. Gödel (1906 - 1978), W. Quine (1908 - 2000)

Truth of mathematical objects

2. There are true mathematical sentences that provide true descriptions of such objects

- Mathematical theorems provide true descriptions of mathematical objects
(Euclid proved that there are infinitely many prime numbers among the positive integers
$\rightarrow$ positive integers are object of study)
- In *set theory* the structures of mathematics are described and studied
- Analysing the nature of various mathematical structures, which are abstract in nature

G. Frege (1848 - 1925), K. Gödel (1906 - 1978), W. Quine (1908 - 2000)

How anyone could acquire knowledge of abstract objects
and recognise their truth?

Mathematical anti-Platonism

Anti-Platonism

Not many tenable alternatives to mathematical Platonism

1. Realistic versions of anti-Platonism: there are things as numbers and sets (and that mathematical theorems are true descriptions), but these things are not abstract objects

2. Mathematical nominalism: reject the belief in the existence of numbers, sets, and so, and reject the belief in the truth of the mathematical theorems (they don't provide a true description of the world).

Realistic anti-Platonism

A. Psychologism

Mathematical theorems are about concrete mental objects of some sort

Popular during the late 19th and early 20th centuries

- Problems of the validity of human knowledge, resolved through the study of the development of mental processes

E. Husserl (1859 - 1938), L. Brouwer (1881 - 1966)

B. Physicalism

Mathematics is about ordinary concrete physical objects of some sort:

- These things exist independently of people and their thoughts

Plato → *redness* (and so numbers) exixts as an abstract object

Aristotle → properties (*accident*) exist in the physical world only; so *redness* only in specific objects; no objects non-spatio-temporal...

D.M. Armstrong (1926 - 2014)

Nominalism

Mathematical objects such as numbers and sets and geometrical objects do not really exist

- Paraphrase nominalists: ordinary sentences (*4 is even*) not interpreted at face value
- Not make straightforward claims about objects, even if there are no such things as numbers, the sentence *4 is even* is still true

D. Hilbert (1862 - 1943), H. Putnam (1926 - 2016)

- Mathematical fictionalists: focused on the *narrative* of the mathematics
- What mathematical sentences (*4 is even*) really mean
- (Agree with the methodology of the platonists) however the mathematical entities do not exist → *4 is even is like 'Santa Claus lives at the North Pole'*

⇒ They are not literally true descriptions of the world, but they are true in a certain well-known story, a fiction

H. Field (b. 1946)

Logicism, intuitionism, and formalism

Logicism

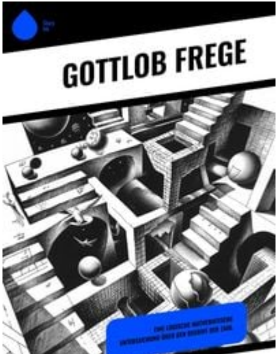
XIX - XX centuries by German mathematician Gottlob Frege and the British mathematician Bertrand Russell

Mathematics is actually logic

Gottlob Frege (1848 - 1925)

In 1879 *Begriffsschrift (Conceptual Notation)* → *symbolic logic* in the XIX century

Philosophy of logic: from a philosophical perspective, the nature and types of logic, including the relation of logic to mathematics
→ In 1884 *Die Grundlagen der Arithmetik (The Foundations of Arithmetic)*
→ In 1893 *Grundgesetze der Arithmetik (Basic Laws of Arithmetic)*



Frege's shipwreck

- On June 16, 1902, a letter from B. Russell (1872 – 1970)
$\rightarrow$ Kindly and correctly, the possibility of deriving a contradiction in Frege's logical system

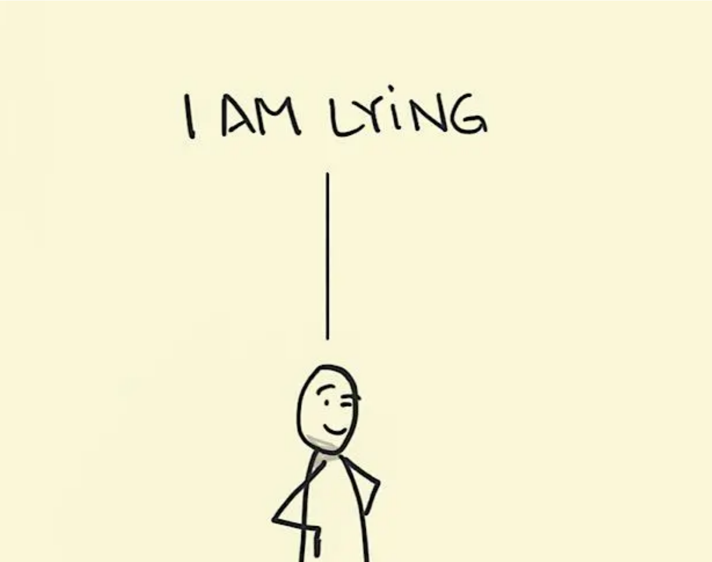
- A logical system employing an unrestricted *comprehension principle*
  - Given any condition expressible by a formula $\phi(x)$
  - It is possible to form the set of all sets x meeting that condition $x|\phi(x)$.
    (the set of all sets - the universal set - would be $x|x = x$.)
  - Russell observed that it allowed the formation of $x|x \notin x$, the set of all non-self-membered sets
  - Is this set - call it R - a member of itself?

Russell's paradox

A - *It is a member of itself* $\Rightarrow$ *it must meet the condition of not being a member of itself*

B - *It is not a member of itself* $\Rightarrow$ *it meets the condition of being a member of itself*

The Liar paradox



Intuitionism

XX century by Dutch mathematician Luitzen Egbertus Jan Brouwer

Primary objects of mathematics are mental constructions with self-evident laws

Intuitionistic logic I

L. Brouwer (1881 - 1966): non-constructive arguments will be avoided if one abandons a principle of classical logic

Principle of the excluded third:

for every proposition p, either p or $\neg p$;

and equivalently: for every p, $\neg\neg p$ implies p

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$\rightarrow$ it may fail in the presence of infinite sets

- Of two numbers x and y in $\mathbb{N}$ one can always decide whether $x = y$ or $x \neq y$

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- Of two numbers x and y in $\mathbb{N}$ one can always decide whether $x = y$ or $x \neq y$
- But of two numbers in $\mathbb{R}$ this may not be possible $\rightarrow$ as one might have to know an infinite number of digits of their decimal expansions

Intuitionistic logic II

For a finite set A, if $\forall x \in A \neg\phi(x)$ leads to a contradiction, $\exists x \in A \phi(x)$
$\Rightarrow$ can be verified by looking at each element of A in turn

The statement *no members of a given set have a certain property* can be disproved by examining in turn each element of the set

For an infinite set A, there is no way in which such an inspection can be carried out

Possible way out $\rightarrow$ moderate form of intuitionism which goes back to Aristotle:
infinite sets do not exist, except potentially

Formalism

XX century by German mathematician David Hilbert

Mathematics set of rules for manipulating formulas
without any reference to the meanings of the formulas

David Hilbert (1862 - 1943)

The theorem of invariants: all invariants can be expressed in terms of a finite number
In 1899 *Grundlagen der Geometrie (The Foundations of Geometry)*
$\rightarrow$ axiomatic treatment of geometry: his definitive set of axioms for Euclidean geometry

$\Rightarrow$ A finite steps of reasoning in logic could not lead to a contradiction
In 1931 Kurt Gödel showed this goal to be unattainable
(*propositions may be formulated that are undecidable; thus, it cannot be known with certainty that mathematical axioms do not lead to contradictions*)

Established the formalistic foundations of mathematics

In 1930 *Naturerkennen und Logik (The Understanding of Nature and Logic)*
Wir mšsen wissen, wir werden wissen (We must know, we shall know)

The collapse of the foundations

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Implicitly raised by D. Hilbert $\rightarrow$ resolved in the negative
- completeness by the Austrian-American logician Kurt Gödel (1906 - 1978)
- decidability by the American logician Alonzo Church (1903 - 1995)

Axiomatic method

An entire system (*e.g.*, a science) is generated in accordance with specified rules by logical deduction from certain basic propositions (*axioms* or *postulates*), which in turn are constructed from a few terms taken as primitive

These terms and axioms may either be arbitrarily defined and constructed or else be conceived according to a model in which some intuitive warrant for their truth is felt to exist

Axiomatized systems:

- Aristotle's syllogistic and Euclid's geometry
- G. Frege, B. Russell and A. Whitehead attempted to formalize all of mathematics in an axiomatic manner

Kurt Gödel

Kurt Gödel (1906–78)

The most important mathematical results of the 20th century

H. Hahn (1879 - 1934), dissertation adviser, one of the leaders of the Vienna Circle
→ However Gödel subscribed to Platonism, theism, mind-body dualism

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No axiomatic theory could possibly capture all arithmetical truths

Gödel's works

- In 1930, doctoral thesis: *Über die Vollständigkeit des Logikkalküls (On the Completeness of the Calculus of Logic)*
Completeness theorem: classical first-order logic is complete: all first-order logical truths can be proved in standard first-order proof systems

- In 1931 the Incompleteness theorems (Theorem VI and Theorem XI): *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme (On Formally Undecidable Propositions of Principia Mathematica and Related Systems)*
$\rightarrow$ *Principia Mathematica* (1910, 1912, and 1913) by A. Whitehead and B. Russell

Impossible an axiomatic mathematical theory that captures all of the truths about the $\mathbb{N}$
$\Rightarrow$ Failure of the projects to construct axiom systems that could be used to prove all mathematical truths

Incompleteness theorems (1931)

Major turning point of XX century logic

Within any axiomatic mathematical system there are propositions that cannot be proved or disproved on the basis of the axioms within that system

The system cannot be simultaneously complete and consistent

No finite set of axioms can produce all possible true mathematical statements
$\rightarrow$ No mechanical (or computer-like) approach will ever be able to exhaust the depths of mathematics

Each system has its own Gödel sentence (G_F). Define a larger system F' containing the whole of F plus G_F as an additional axiom $\rightarrow$ not a complete system, because Gödel's theorem will also apply to F'. In this case, G_F is indeed a theorem in F' (no contradiction is presented by its provability within F'). And a new Gödel statement $G_{F'}$ for F', showing that F' is also incomplete.

G_F: true but unprovable

Since the Gödel sentence cannot itself formally specify its intended interpretation, the truth of the sentence G_F may only be arrived at via a meta-analysis from outside the system

The Gödel sentence of a consistent theory is true as a statement about the intended interpretation of arithmetic, but will be false in some nonstandard models of arithmetic as a consequence of F

A Gödel sentence (G_F) for a system F makes a similar assertion to the liar sentence, but with truth replaced by provability: G_F says G_F is not provable in the system F
The analysis of the truth and provability of G_F is a formalized version of the analysis of the truth of the liar sentence

Second incompleteness theorem: Theorem XI

- For each formal system F containing basic arithmetic, it is possible to canonically define a formula $Cons(F)$ expressing the consistency of F
$\rightarrow$ The property that there does not exist a natural number coding a formal derivation within the system F whose conclusion is a syntactic contradiction.
($0 = 1$, in which case $Cons(F)$ states *there is no natural number that codes a derivation of 0 = 1 from the axioms of F*)

This canonical consistency statement $Cons(F)$ will not be provable in F

For any consistent system F
within which a certain amount of elementary arithmetic can be carried out
the consistency of F cannot be proved in F itself

Gödel's Platonism

The incompleteness theorem $\Rightarrow$ unprovable mathematical truths
Would go a long way toward establishing Platonism

Mathematical truth is objective - beyond mere human provability or human axiom systems

In 1964 *What Is Cantor's Continuum Problem?*, contrast the objection:
- Humans acquire all knowledge of the external world through sensory perception
- Platonism asserts that mathematical objects, are nonphysical objects, cannot be perceived by the senses
- *In addition to the normal five senses, humans also possess a faculty of mathematical intuition* $\rightarrow$ *people to grasp the nature of numbers or to see them in the mind's eye*

This faculty exists outside of space and time

Conclusion

Conclusion?

An Unended Open Quest!

Backup

The significance of the choice

People who voted $N = 23$ on $n = 3$ possible choices.
Number of Degree of freedom $ndf = (\text{number of choices}) - 1 = 2$.
Observed count $O = [5, 8, 10]$.
The significant level $\alpha = 0.05$

1. Null Hypothesis H_0. The votes are evenly distributed across the three options
   Expected value:
   $$E = \frac{N}{n} = \frac{23}{3} \sim 7.7$$
2. χ^2 test to determine whether the observed distribution of votes significantly differs from an expected distribution
   $$\chi^2 = \sum \frac{(O - E)^2}{E} = 1.65$$
3. Using a χ^2 table, the p-value for $\chi^2 = 1.65$ with $ndf = 2$ is ~ 0.44

New foundations

- Early Greek philosophy $\rightarrow$ dispute as to which is more basic, arithmetic or geometry
  - Pythagorean: originally only natural and rational numbers exist; then accepted the discovery that $\sqrt{2}$ (ratio of the diagonal of a square to its side), not be expressed as the ratio of whole numbers
  - Axiomatic method and notion of proof: Plato's Academy and Euclid's Elements
    - A number of basic mathematical truths (*axioms* or *postulates*)
    - Other true statements can be derived in a finite number of steps (*theorems*)

    In principle, a mechanical device, a computer, can generate all theorems

$\Rightarrow$ Beginning of the XX century $\rightarrow$ rebuild mathematics on new bases, independent of *geometric intuitions* or *self evidences*

Foundations of Mathematics

- Study of the logical and philosophical basis of mathematics

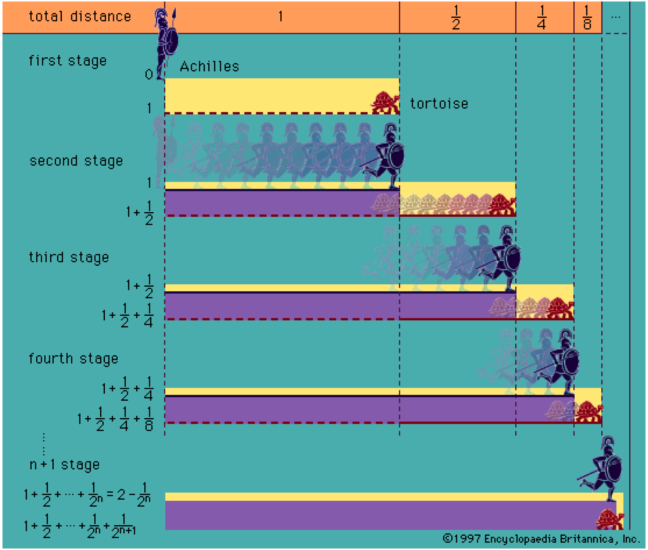
$\rightarrow$ Whether the axioms of a given system ensure its completeness and its consistency

Mathematics as a model for rational inquiry, extensively used in the sciences, foundational studies have far-reaching consequences for the reliability and extensibility of rational thought itself

Euclid's *Elements* ($\sim$300 BCE) $\rightarrow$ 2,000 years of mathematics perfectly solid
A set of formal logical arguments based on a few basic terms and axioms
$\rightarrow$ a systematic method of rational exploration well into the XIX century

- Alternative geometries $\Rightarrow$ Euclidean geometry not correspond with *reality*
- G. Cantor (1845 - 1918) $\Rightarrow$ set theory to answer paradoxical results and provide a more rigorous foundation for mathematics

Achille's paradox



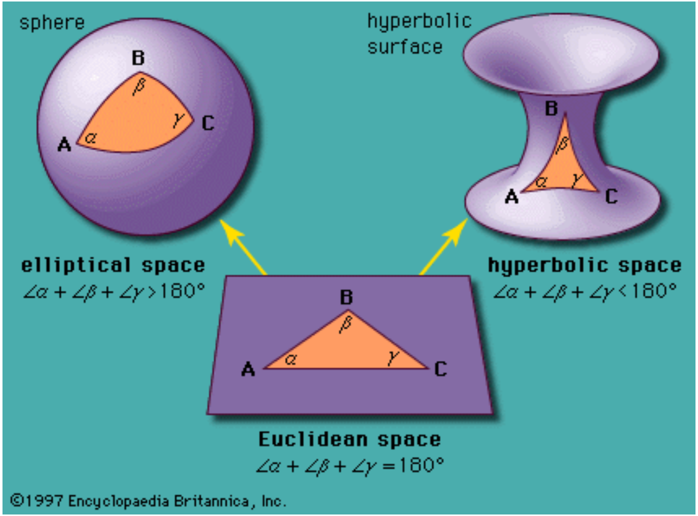
Non-Euclidean Geometries I

- Euclid's axiomatic treatment of geometry
- The V postulate, less obvious or fundamental than the others: there is exactly one parallel to a given line through a given point (as it is formulated today)

- V postulate is independent of the others (XIX century)

Euclidean Geometry not the only *true* geometry but only one of a number of possible geometries

Non-Euclidean Geometries II



Nontraditional versions of Mathematical Platonism

I) P. Maddy (b. 1950): mathematics is about abstract objects, nonphysical and nonmental, though they are located in space and time
$\Rightarrow$ a set of physical objects, right where the physical objects themselves are located
- there are infinitely many sets

II) M. Balaguer and E. Zalta: not just the existence of abstract objects but the existence of as many abstract objects as there can possibly be
full-blooded Platonism
$\rightarrow$ can explain how humans could acquire knowledge of abstract objects

III) M. Resnik (b. 1938) and S. Shapiro (b. 1951): the real objects of study in mathematics are structures, or patterns
Structuralism
- infinite series, geometric spaces, and set-theoretic hierarchies
- individual mathematical objects (the number 4) not really objects $\rightarrow$ simply positions in structures, or patterns
(the number 4, for instance, is just the fourth position in the positive integer pattern)

Clarification

The viewpoints discussed above → what the sentences of mathematics actually say and what they are actually talking about

- Intellectual climate in which they were developed

- For every concept exists a set of things that fall under that concept
(Even concepts such as 'mermaid' are associated with a set: the empty set)
- Russel: *there is no set corresponding to the concept 'not a member of itself'*
  - Suppose that there were such a set a set of all the sets that are not members of themselves, S
  - Is S a member of itself?
    - If it is, then it is not (because all the sets in S are not members of themselves)
    - If S is not a member of itself, then it is (because all the sets not in S are members of themselves)
  - Either way, a contradiction follows. Thus, there is no such set as S

Mathematical assertion and denial

A. *The object O has the property P*
→ There is a proof that the object O has the property P
B. *not-P*
→ A contradiction can be proven from P

⇒ Reject the traditionally claim that for any mathematical sentence P, either P or not-P is true
- They reject *non-constructive proofs*:
Irrational numbers a and b can produce a^b rational;
→ If $\sqrt{2}^{\sqrt{2}}$ is rational it is proved
→ Otherwise take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, so that $a^b = 2$

⇒ The argument is non-constructive, it does not tell which alternative holds
The result can be proved constructively by taking $a = \sqrt{2}$ and $b = 2log_2 3$
But there are other classical theorems for which no constructive proof exists

Consistency and effectivness

A theory is a set of well-formed formulas with no free variables
A theory is consistent if there is no formula F such that both F and its negation are provable. ω-consistency is a stronger property: suppose that $F(x)$ is a formula with one free variable x. In order to be ω-consistent, the theory cannot prove both $\exists\, m\ F(m)$ while also proving $\neg\ F(n)$ for each natural number n. The theory is assumed to be effective,

which means that the set of axioms must be recursively enumerable. This means that it is theoretically possible to write a finite-length computer program that, if allowed to run forever, would output the axioms of the theory (necessarily including every well-formed instance of the axiom schema of induction) one at a time and not output anything else. This requirement is necessary; there are theories that are complete, consistent, and include elementary arithmetic, but no such theory can be effective.

First incompleteness theorem (1931)
Theorem VI

A major turning point of 20th-century logic
- No finite set of axioms can be devised that will produce all possible true mathematical statements
$\rightarrow$ No mechanical (or computer-like) approach will ever be able to exhaust the depths of mathematics

If some particular statement is undecidable within a given formal system, it may be incorporated in another formal system as an axiom or be derived from the addition of other axioms

Theorem VI: First Incompleteness Theorem

Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; *i.e.* there are statements of the language of F which can neither be proved nor disproved in F

Raatikainen 2020

Each effectively generated system has its own Gödel sentence (G_F). It is possible to define a larger system F' that contains the whole of F plus G_F as an additional axiom. This will not result in a complete system, because Gödel's theorem will also apply to F', and thus F' also cannot be complete In this case, G_F is indeed a theorem in F', because it is an axiom. Because G_F states only that it is not provable in F, no contradiction is presented by its provability within F'. However, because the incompleteness theorem applies to F', there will be a new Gödel statement $G_{F'}$ for F', showing that F' is also incomplete. $G_{F'}$ will differ from G_F in that $G_{F'}$ will refer to F', rather than F

True but unprovable

The sentence G_F is often said to be *true but unprovable*
Since the Gödel sentence cannot itself formally specify its intended interpretation, the truth of the sentence G_F may only be arrived at via a meta-analysis from outside the system

This meta-analysis can be carried out within the weak formal system known as primitive recursive arithmetic, which proves the implication $Con(F) \to G_F$, where $Con(F)$ is a canonical sentence asserting the consistency of F

The Gödel sentence of a consistent theory is true as a statement about the intended interpretation of arithmetic, the Gödel sentence will be false in some nonstandard models of arithmetic as a consequence of *Gödel's completeness theorem* $\to$ when a sentence is independent of a theory, the theory will have models in which the sentence is true and models in which the sentence is false

This sentence is false

An analysis of the liar sentence shows that it cannot be true (for then, as it asserts, it is false), nor can it be false (for then, it is true)

A Gödel sentence G for a system F makes a similar assertion to the liar sentence, but with truth replaced by provability: G says *G is not provable in the system F* The analysis of the truth and provability of G is a formalized version of the analysis of the truth of the liar sentence

It is not possible to replace not provable with false in a Gödel sentence because the predicate *Q is the Gödel number of a false formula* cannot be represented as a formula of arithmetic

Symbols

- A constant symbol 0 for zero
- A unary function symbol S for the successor operation and two binary function symbols $+$ and $\times$ for addition and multiplication
- Three symbols for logical conjunction, $\wedge$, disjunction, $\vee$, and negation, $\neg$
- Two symbols for universal, $\forall$, and existential, $\exists$, quantifiers
- Two symbols for binary relations, $=$ and $<$, for equality and order (less than)
- Two symbols for left, (and right,) parentheses for establishing precedence of quantifiers
- A single variable symbol, x and a distinguishing symbol * that can be used to construct additional variables of the form x^*, x^{**}, x^{***}, ...

Second incompleteness theorem Theorem XI

For each formal system F containing basic arithmetic, it is possible to canonically define a formula $Cons(F)$ expressing the consistency of F. This formula expresses the property that *there does not exist a natural number coding a formal derivation within the system F whose conclusion is a syntactic contradiction*.
The syntactic contradiction is often taken to be $0 = 1$, in which case $Cons(F)$ states *there is no natural number that codes a derivation of '0 = 1' from the axioms of F*

This canonical consistency statement $Cons(F)$ will not be provable in F

For any consistent system F within which a certain amount of elementary arithmetic can be carried out, the consistency of F cannot be proved in F itself. This theorem is stronger than the first incompleteness theorem because the statement constructed in the first incompleteness theorem does not directly express the consistency of the system. The proof of the second incompleteness theorem is obtained by formalizing the proof of the first incompleteness theorem within the system F itself.

Expressing consistency

There are many ways to express the consistency of a system, and not all of them lead to the same result. The formula $Cons(F)$ from the second incompleteness theorem is a particular expression of consistency

Other formalizations of the claim that F is consistent may be inequivalent in F, and some may even be provable

For example, first-order Peano arithmetic (PA) can prove that *the largest consistent subset of PA* is consistent. But, because PA is consistent, the largest consistent subset of PA is just PA, so in this sense PA *proves that it is consistent*

What PA does not prove is that the largest consistent subset of PA is, in fact, the whole of PA

(The term *largest consistent subset of PA* is meant here to be the largest consistent initial segment of the axioms of PA under some particular effective enumeration)

There is no hope of proving, for example, the consistency of PA using any finitistic means that can be formalized in a system the consistency of which is provable in PA