**I. Mathematics from a Philosophical Perspective**

Mathematics has always been central to knowledge, but what is its true nature? Is it discovered or invented?

1. **Ancient Foundations**

• Greek mathematics: axiomatic approach (Euclid)

• Pythagoreans: numbers as fundamental, but struggled with irrational numbers

• The role of proofs: Mathematics became more formalized through logical deductions

2. **Mathematics as the “Queen of Sciences”**

• Philosophers like **Descartes, Spinoza, and Comte** sought to model knowledge on mathematical reasoning

• **Galileo**: “The universe is written in mathematical language” → But why does math describe reality so well?

**II. Kant’s Epistemological Challenge**

• Kant redefined mathematics: not a reflection of ultimate reality but a **structural framework imposed by the mind**.

• **Phenomena vs. Noumena**: We perceive the world through mathematical structures, but this doesn’t mean math describes reality itself.

• Mathematics is **universal** for human cognition but limited to our experience of the world.

**III. Philosophy of Mathematics: Key Questions**

1. **What do mathematical statements mean?** (Semantic theory)

2. **Do mathematical objects exist independently?** (Ontology)

3. **Are mathematical truths discovered or created?**

**IV. Platonism vs. Anti-Platonism**

1. **Mathematical Platonism**

• Mathematical objects (e.g., numbers, sets) **exist independently** of human thought.

• They are **non-physical, eternal, unchanging**, and discovered rather than invented.

• Supported by thinkers like **Plato, Frege, and Gödel**.

2. **Challenges to Platonism**

• How do we access mathematical truths if they exist beyond space and time?

• If math is independent, why does it correspond so well to physical reality?

3. **Mathematical Anti-Platonism**

• Rejects independent mathematical existence

• Includes **formalism (math as rules), intuitionism (math as mental constructs), and nominalism (math as language/convention)**.

**V. The 19th-20th Century Crisis in Mathematics**

1. **The Challenge of Non-Euclidean Geometry**

• Euclid’s 5th postulate (parallel lines) questioned → Alternative geometries (Gauss, Lobachevsky, Riemann) emerged.

• If multiple valid geometries exist, does math describe reality or just provide models?

2. **Set Theory & Paradoxes** (Cantor, Russell)

• **Cantor’s infinity** introduced paradoxes (e.g., Russell’s Paradox).

• Can we ensure **mathematical consistency**?

3. **Hilbert’s Formalism & Gödel’s Incompleteness Theorems**

• **Hilbert:** Math should be based on complete, consistent axioms.

• **Gödel:** Proved this is **impossible**—no system can be both complete & consistent.

In the late 19th and early 20th centuries, mathematics faced a foundational crisis, leading to the emergence of three significant schools of thought: **Logicism, Intuitionism, and Formalism**. This crisis arose due to the discovery of non-Euclidean geometries and concerns about the completeness and consistency of mathematical systems. Mathematicians sought to establish a solid foundation for mathematics, leading to extensive philosophical and logical investigations.

**Logicism**

**Key Figures**: Gottlob Frege, Bertrand Russell

Logicism asserts that mathematics is reducible to logic, meaning that mathematical truths are ultimately logical truths.

* **Frege’s Contributions**:
  + Developed symbolic logic to express mathematical statements formally.
  + Published works such as *The Foundations of Arithmetic* (1884) and *Basic Laws of Arithmetic* (1893).
  + His system aimed to derive all of mathematics from logical axioms.
* **Russell’s Paradox (1902)**:
  + Bertrand Russell identified a contradiction in Frege’s system.
  + The paradox arises when considering the set of all sets that do not contain themselves:
    - If such a set belongs to itself, it contradicts its definition.
    - If it does not belong to itself, it should belong to the set.
  + This paradox undermined Frege’s program, leading to his eventual abandonment of logicism.

**Intuitionism**

**Key Figure**: L.E.J. Brouwer

Intuitionism holds that mathematical objects are mental constructions rather than external realities. The approach emphasizes constructive proofs and rejects non-constructive arguments.

* **Core Principles**:
  + Mathematics is based on self-evident laws of thought.
  + The **Law of the Excluded Middle** (*either P or not P*) is rejected for infinite sets.
  + Infinite sets exist only **potentially**, not as completed entities (influenced by Aristotle’s distinction between actuality and potentiality).
* **Challenges**:
  + Works well for **finite sets**, where elements can be verified individually.
  + Struggles with **infinite sets**, as not all elements can be explicitly constructed or verified.

**Formalism**

**Key Figure**: David Hilbert

Formalism views mathematics as a symbolic game, independent of meaning or external reality. Mathematical statements follow a system of rules and manipulations.

* **Hilbert’s Program**:
  + Aimed to formalize all of mathematics within a consistent and complete axiomatic system.
  + Published *Foundations of Geometry* (1899), revising Euclidean geometry’s axioms.
  + Believed that formal systems could avoid contradictions and inconsistencies.
* **Gödel’s Incompleteness Theorems (1931)**:
  + Kurt Gödel proved that:
    1. In any **consistent** formal system powerful enough to express arithmetic, there exist **true** mathematical statements that cannot be proven within the system.
    2. A system’s consistency cannot be proven using only its own axioms.
  + This shattered Hilbert’s dream of a complete, contradiction-free foundation for mathematics.

**Kurt Gödel: The Most Important Mathematician of the 20th Century**

Kurt Gödel stands as one of the most influential mathematicians and logicians of the 20th century. His groundbreaking work, particularly his Incompleteness Theorems, reshaped our understanding of mathematical logic and the limits of formal systems.

**Early Life and Influences**

Gödel was a student of Hans Hahn, one of the founding members of the Vienna Circle, a group deeply invested in Logical Positivism. Logical Positivism, or Neopositivism, asserts that only statements that can be empirically verified are meaningful. However, Gödel fundamentally opposed this philosophy, distancing himself from the Vienna Circle's views. Instead, he subscribed to mathematical Platonism—the belief that mathematical objects exist independently of human thought.

**Completeness Theorem (1930)**

Gödel's doctoral thesis, "On the Completeness of the Calculus of Logic," established that first-order classical logic is complete. This means that if a statement is true in all models of a logical system, it can be proven within that system. While this was an impressive achievement, it did not cause much upheaval in the mathematical world.

**Incompleteness Theorems (1931)**

Gödel’s real revolution came in 1931 when he published his **Incompleteness Theorems**, proving fundamental limitations in formal mathematical systems.

1. **First Incompleteness Theorem:** Within any consistent formal system that is capable of expressing elementary arithmetic, there exist true statements that cannot be proven within that system.
2. **Second Incompleteness Theorem:** No such formal system can prove its own consistency.

These theorems dismantled the belief, held by mathematicians such as Hilbert and the authors of *Principia Mathematica*(Russell and Whitehead), that mathematics could be completely formalized using a finite set of axioms.

**Implications and Legacy**

Gödel’s results had profound consequences:

* They demonstrated the inherent limitations of formal mathematical systems.
* They showed that mathematics is not reducible to pure logic, refuting the core premise of Logicism.
* They suggested that mathematical truth extends beyond formal proof, aligning with a Platonic view of mathematics.

**Philosophical and Theological Views**

Gödel was a Platonist, believing in the objective reality of mathematical truths. He also embraced **theism**, asserting that an ultimate being or prime mover was necessary to halt the infinite regress of axiomatic extensions. Later in life, he proposed that humans have a "mathematical intuition" akin to a "mind’s eye," enabling us to grasp mathematical truths beyond formal proof.

**Conclusion**

Gödel’s work irrevocably altered mathematics, logic, and even artificial intelligence. His Incompleteness Theorems revealed the fundamental constraints of formal systems, emphasizing the necessity of an open-ended approach to knowledge. While his mathematical discoveries remain at the heart of modern logic, his philosophical and theological insights continue to inspire debate across multiple disciplines.

My personal conclusion and the wish for each one of us is to be able continuing having an open endless quest, in our lives.