FROM IMPOSSIBLE TO DOABLE: COMPLETE FUNCTION SPACE FOR TWO-LOOP SIX-POINT SCATTERING AMPLITUDES

Based on 2403.19742 and 2501.01847 with J. Henn, J. Miczajka, T. Peraro, Y. Xu, Y. Zhang
&
25XX.XXXXX with J. Miczajka

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STATE-OF-THE-ART & MOTIVATION

- # Loops + # legs = 7
  - Three-loop four-point [Henn, Smirnov, Smirnov ’13; Di Vita, Mastrolia, Schubert, Yundin ’14; Canko, Syrrakos ’21 & ’22; Henn, Lim, Torres Bobadilla ’23; Gehrman, Henn, Jakubčik, Lim, Mella, Syrrakos, Tancredi, Torres Bobadilla ’24]
  - Two-loop five-point [Gehrmann, Henn, Lo Presti ’18; Abreu, Dixon, Herrmann, Page, Zeng ’18; Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia ’18; Abreu, Ita, Moriello, Page, Tschernow, Zeng ’20; Chicherin, Sotnikov, ’20; Abreu, Ita, Page, Tschernow ’21; Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia ’23; Badger, Becchetti, Cahubey, Marzucca ’23; Febres Cordero, Figueiredo, Kraus, Page, Reina ’23, Abreu, Chicherin, Sotnikov, Zoia ’24, Becchetti, Dlapa, Zoia ’25]
- First results for three-loop five-point [Liu, AM, Miczajka, Xu, Xu, Zhang ’24]

State-of-the-art & Motivation

- # Loops + # legs = 7
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- First results for three-loop five-point [Liu, AM, Miczajka, Xu, Xu, Zhang ’24]
- Very little is known about two-loop six-point processes in general theories
- Phenomenologically interesting: necessary ingredient for amplitudes computations
- Theoretically interesting: Analytic structure of QCD function space

OUTLINE

I. Function space for planar two-loop six-point scattering amplitudes

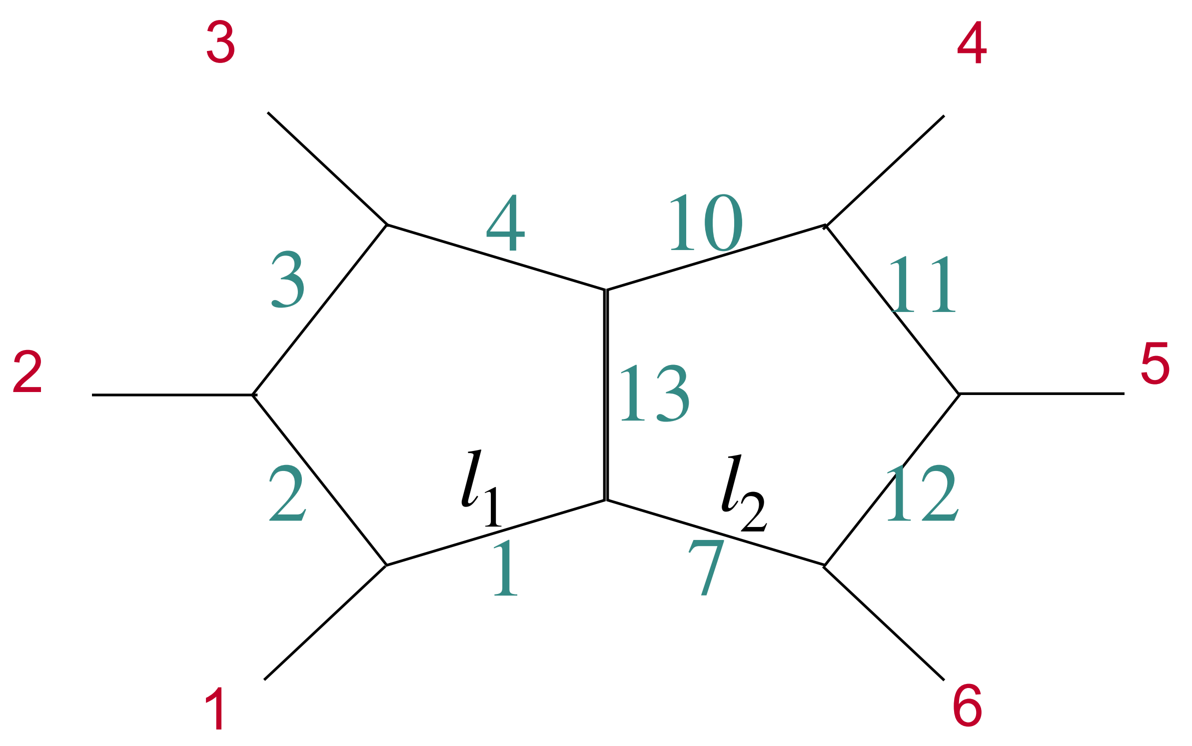
II. How to predict the alphabet of polylogarithmic integrals without obtaining differential equations?

I. Two-loop six-point Feynman integrals

Notation & kinematics

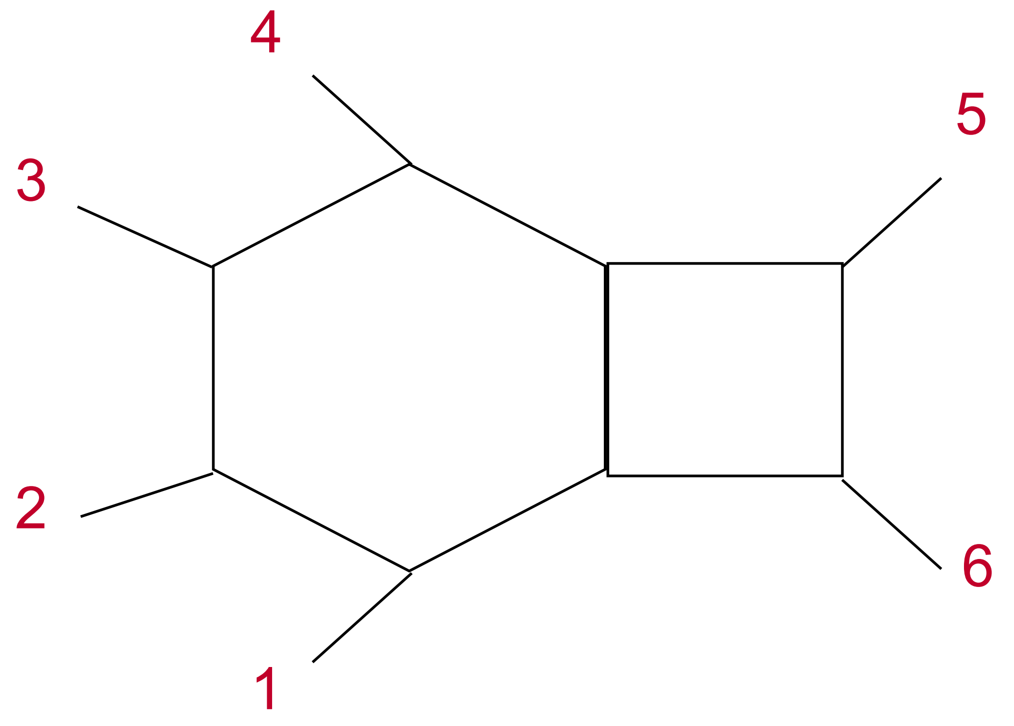
$$I[a_1,\ldots,a_{13}] = e^{2\epsilon\gamma_E}\int \frac{\mathrm{d}^{d_0-2\epsilon}l_1\mathrm{d}^{d_0-2\epsilon}l_2}{-\pi^{(d_0-2\epsilon)/4}}\frac{1}{D_1^{a_1}\ldots D_{13}^{a_{13}}}$$

$$D_1 = l_1^2,\ D_2 = (l_1+p_1)^2,\ \ldots,\ D_{13} = (l_1+l_2)^2 \qquad a_i \in \mathbb{Z}$$



External momenta:

$$p_i^2 = 0, \quad i = 1,\ldots,6 \qquad \sum_{i=1}^{6} p_i = 0 \qquad p_i \in \mathbb{R}^{D_{ext}}$$



Kinematics:

$$\vec{v} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345}\}$$

$$s_{ij} = (p_i+p_j)^2,\ s_{ijk} = (p_i+p_j+p_k)^2$$

$$G := det(p_i\cdot p_j) = 0, \quad i,j \in \{1,\ldots,5\}$$

Solved by a momentum twistor parametrization

CANONICAL DIFFERENTIAL EQUATIONS

- Integrals within the integral satisfy integration-by-part (IBP) identities
    - Family is spanned by a finite number of basis integrals (MI)
    - Can find a particular basis such that:

[Chetyrkin, Tkachov '81; Laporta '00]
[Smirnov, Petukhov '10]

$$\mathrm{d}\vec{I}(\vec{v},\epsilon) = \epsilon \left[\sum_{i,j} A_i \mathrm{dlog}(W_j(\vec{v})) \right] \vec{I}(\vec{v},\epsilon)$$

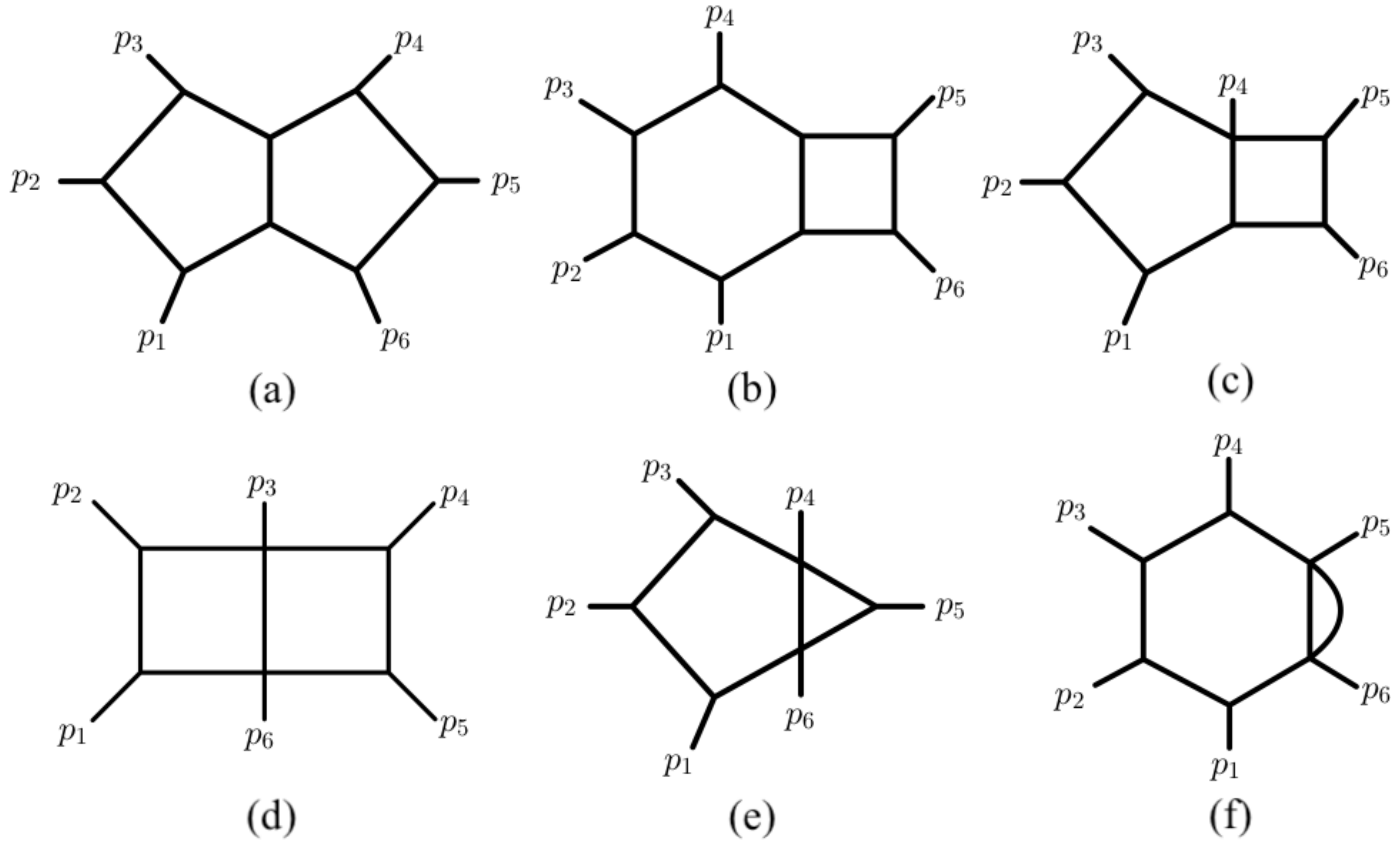
[Henn '13]

vector of N master integrals

constnat N x N matrices

symbol letters

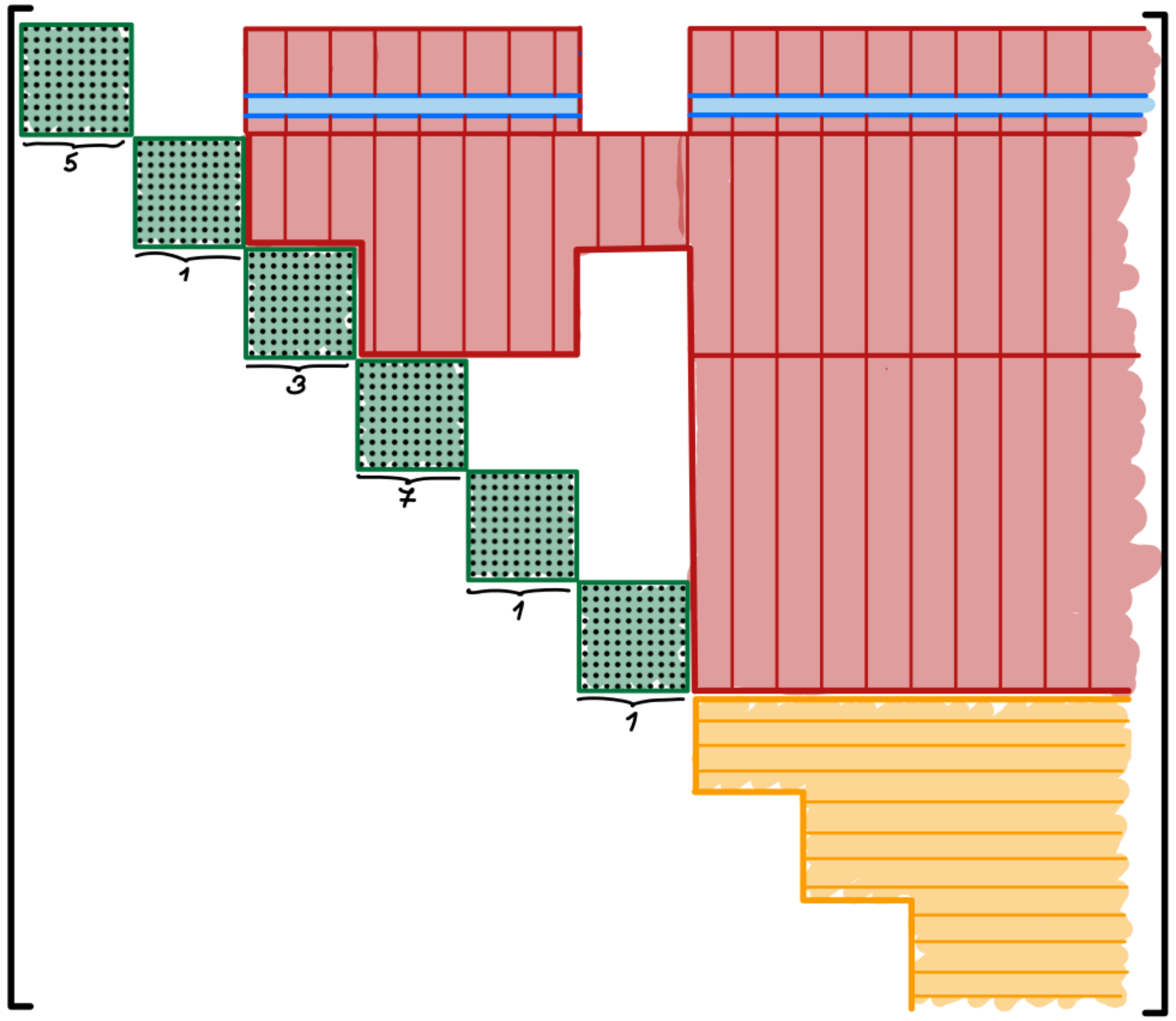
PLANAR TWO-LOOP INTEGRAL FAMILIES



Family	# MI top sector	# MI total
(a) dp	5	267
(b) hb	1	202
(c) pb	3	117
(d) db	7	66
(e) pt	1	45
(f) hbb	1	32

DIFFERENTIAL EQUATION BLOCKS IN CANONICAL FORM

$$
\mathrm{d}\begin{bmatrix} I_{\mathrm{dp}} \\ I_{\mathrm{hb}} \\ I_{\mathrm{pb}} \\ I_{\mathrm{db}} \\ I_{\mathrm{pt}} \\ I_{\mathrm{hbb}} \\ I_{\mathrm{sub}} \\ \vdots \end{bmatrix} = \epsilon\, \mathrm{d}
$$



$$
\begin{bmatrix} I_{\mathrm{dp}} \\ I_{\mathrm{hb}} \\ I_{\mathrm{pb}} \\ I_{\mathrm{db}} \\ I_{\mathrm{pt}} \\ I_{\mathrm{hbb}} \\ I_{\mathrm{sub}} \\ \vdots \end{bmatrix}
$$

- Six-particle blocks on the maximal cut — [Henn, Peraro, Xu, Zhang '21]
- Five-particle integrals with one off-shell leg — [Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]
- Off-diagonal blocks* — [Henn, AM, Miczajka, Peraro, Xu, Zhang '24, '25; AM '24]
- To be reconstructed

Reduces computational time from $\mathcal{O}$(years) → $\mathcal{O}(2$ days)

- Conjectured alphabet + UT basis + finite fields ⇒ major speed up

We use `FiniteFlow` [Peraro '19]

* hb family independently computed in [Abreu, Monni, Page, Usovitsch '24]

Double pentagon function space

- Up to 5 new functions contribute to the finite part of scattering amplitudes
  - $I_4^{DP} \sim \epsilon^6$, I_1^{DP} and I_3^{DP} are evanescent
  - I_2^{DP} and I_5^{DP} known from $\mathcal{N} = 4$ sYM

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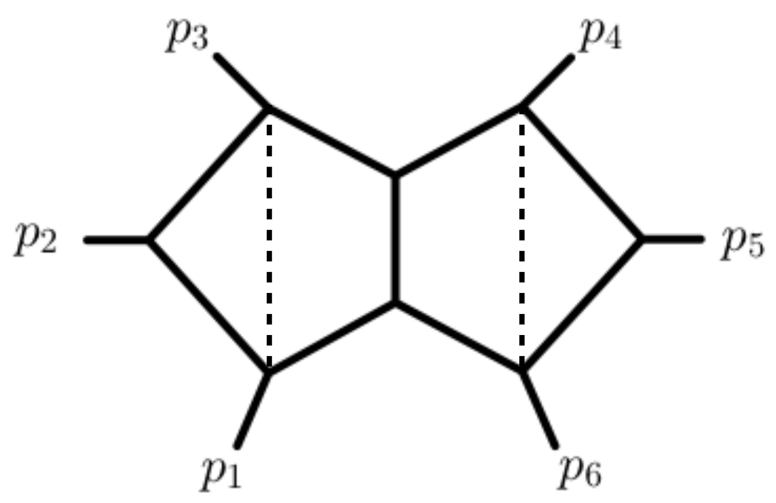
$$I_2^{DP} = -2\tilde{\Omega}^{(2)}_{odd} + \mathcal{O}(\epsilon)$$

$$I_5^{DP} = 2\Omega^{(2)}_{even} + \mathcal{O}(\epsilon)$$

$\Omega^{(2)} =$



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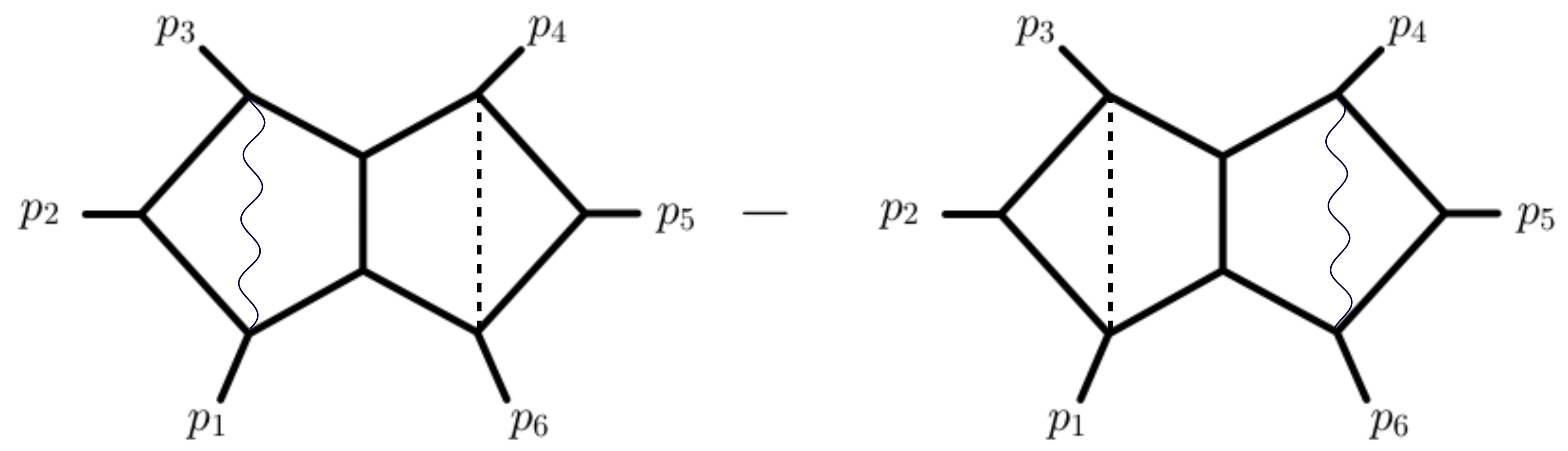

[Arkani-Hamed, Bourjaily, Cachazo, Trnka ’10; Dixon, Drummond, Henn ’11]

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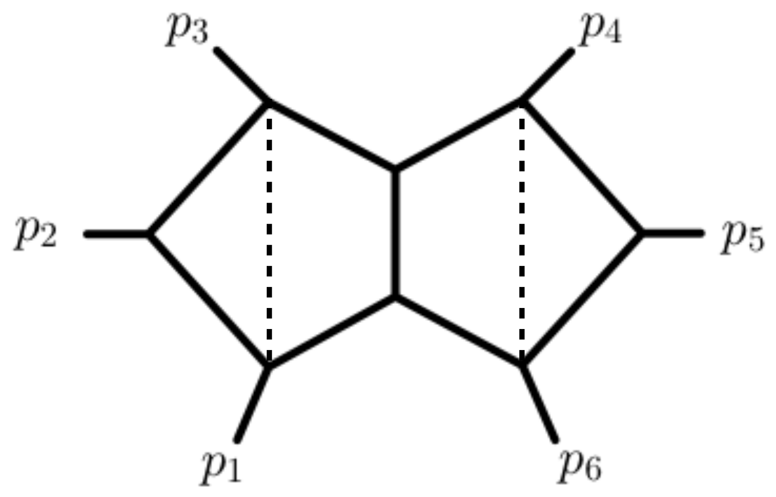

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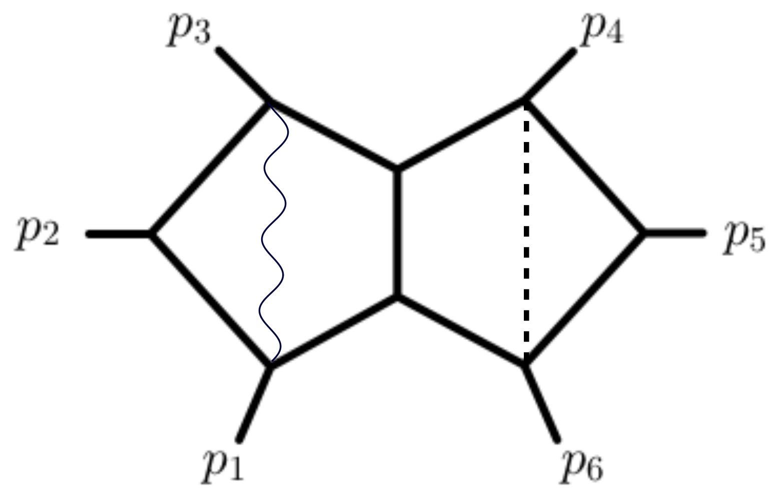

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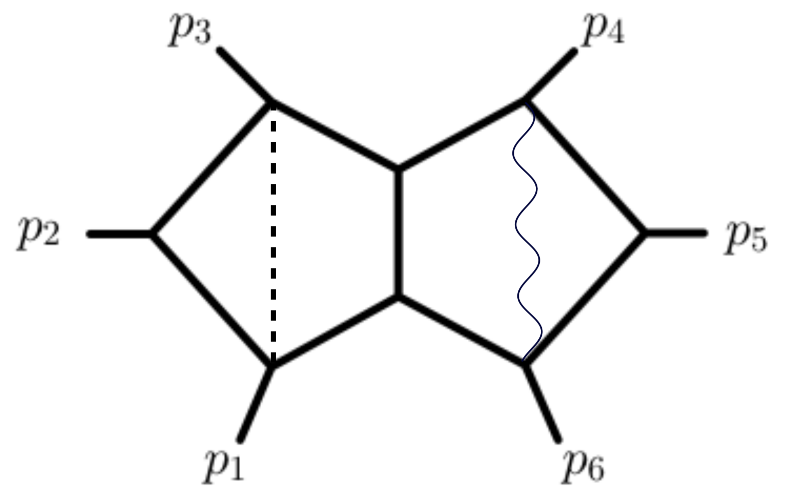
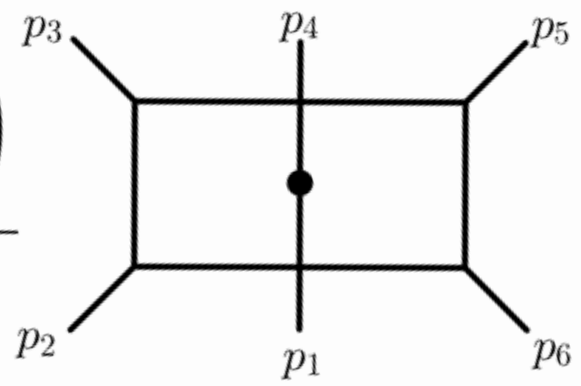


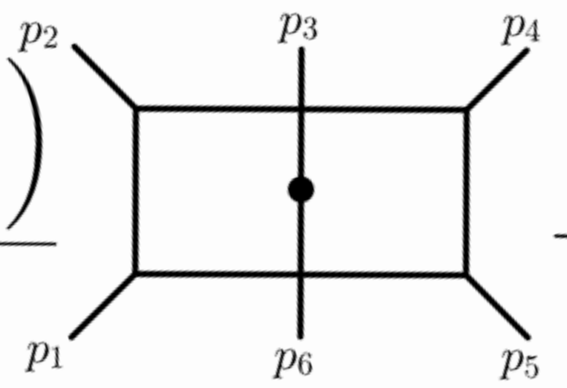
[Arkani-Hamed, Bourjaily, Cachazo, Trnka ’10; Dixon, Drummond, Henn ’11]

- Both can be further related to subsector integrals

$$\tilde{\Omega}^{(2)}_{odd} = \frac{\Delta_6 G\begin{pmatrix} l_2 - p_5 - p_6 & p_5 & p_6 & p_2 & p_3 \\ l_1 + p_5 + p_6 & p_5 & p_6 & p_2 & p_3 \end{pmatrix}}{4G(p_5, p_6, p_2, p_3)}$$



$$- \frac{\Delta_6 G\begin{pmatrix} l_2 - p_6 & p_5 & p_4 & p_2 & p_1 \\ l_1 + p_6 & p_5 & p_4 & p_2 & p_1 \end{pmatrix}}{4G(p_5, p_4, p_2, p_1)}$$



$$+\mathcal{O}(\epsilon)$$

Solving canonical differential equations

- Laurent expansion around $\epsilon = 0$

$$\vec{I}(\vec{v},\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \vec{I}^{(k)}(\vec{v})$$

- We can solve differential equations order by order in ϵ

$$\vec{I}^{(k)}(\vec{v}) = \vec{I}^{(k)}(\vec{v}_0) + \int_\gamma \mathrm{d}\tilde{A}(\vec{v}')\vec{I}^{(k-1)}(\vec{v}')$$

boundary vector

$$\sum_{i,j} A_i \mathrm{dlog}(W_j(\vec{v}))$$

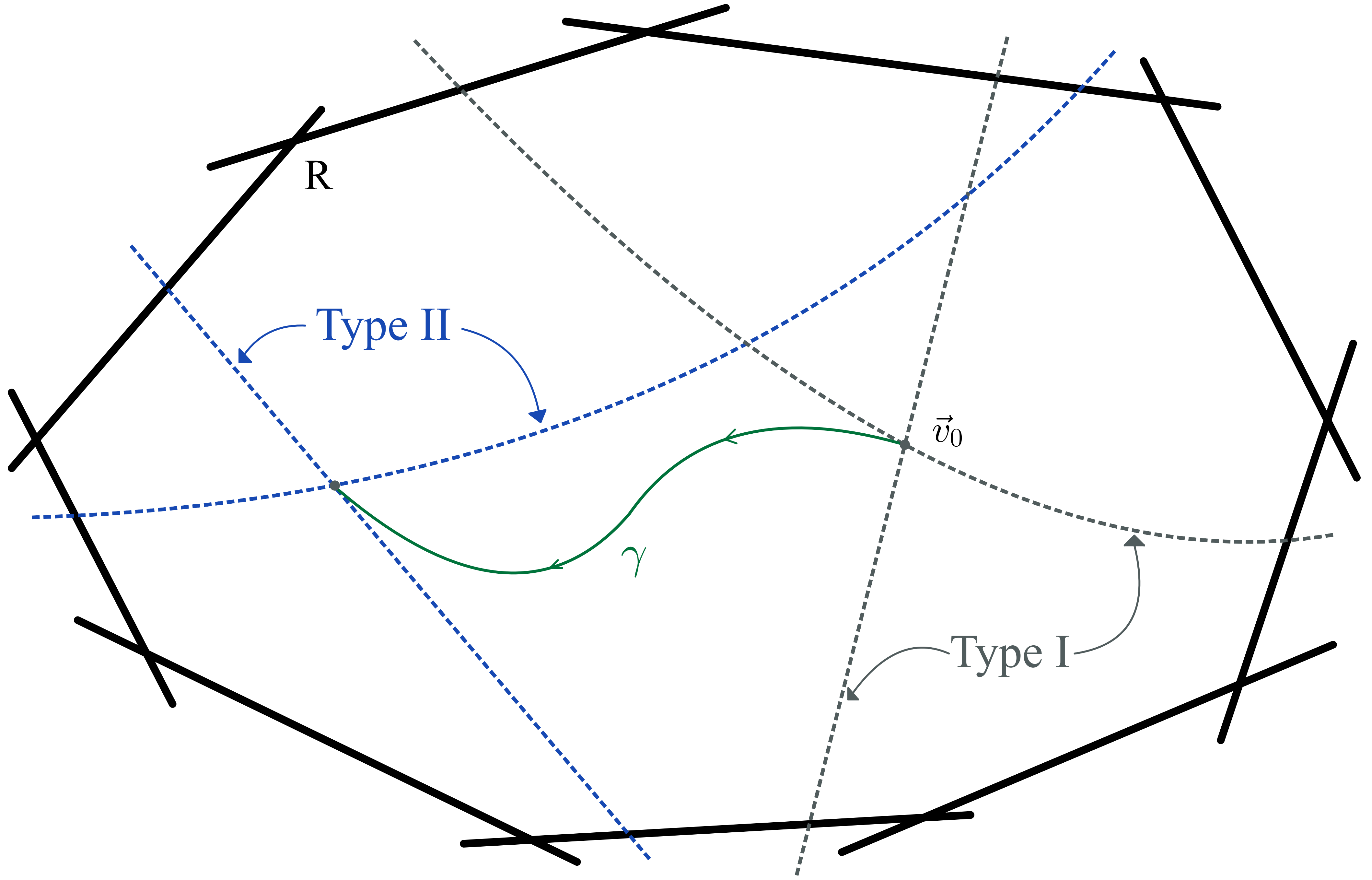
Classification of function space

- The symbol:

$$\mathcal{S}\left(\left[W_1,\ldots,W_k\right]_{\vec{v}_0}(\vec{v})\right) = W_1 \otimes \cdots \otimes W_k$$

Weight	1	2	3	4
# full function space	9	62	319	945
# (one-loop)^2	9	59	221	428
# two-loop five-point	9	59	263	594
# genuinely two-loop six-point	-	-	3	45

DETERMINING BOUNDARY VALUES



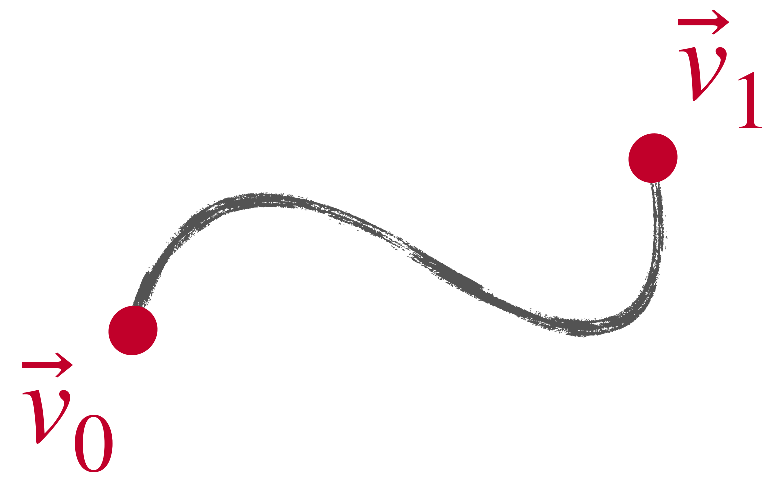
- Analytic value of basis integrals at a single point in Euclidean region

$$\vec{v}_0 = \{-1, -1, -1, -1, -1, -1, -1, -1, -1\}$$

- Absence of certain spurious singularities and by matching to a single sunrise integral
- Linear combinations of polylogarithms at the sixth root of unity

[Henn, Smirnov, Smirnov '15]

SOLUTIONS TO DE



- Goal: Express it in terms of known functions

- It might be possible to express the result in terms of classical polylogarithms, but most likely inefficient for evaluations!

- Instead, use one-fold integral representation along some path! [Caron-Huot, Henn '14]

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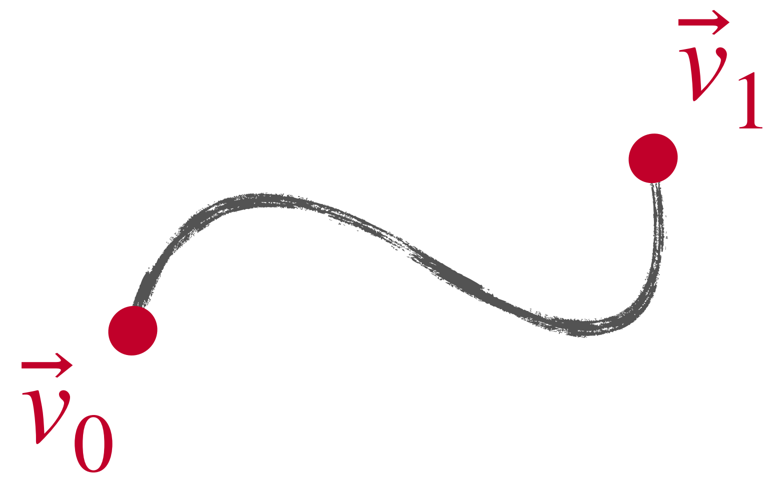

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- Instead, use one-fold integral representation along some path! [Caron-Huot, Henn '14]

Step 1: Analytically solve the DE up to weight 2 in terms of $\{\log, \log^2, \mathrm{Li}_2\}$

Step 2: Write weight 3, 4 solution as integrals over weight 2:

$$\vec{I}^{(3)}(\vec{v}_1) = \vec{I}^{(3)}(\vec{v}_0) + \int_0^1 \mathrm{d}t \, \frac{\partial A}{\partial t} \cdot \vec{I}^{(2)}(t)$$

$$\vec{I}^{(4)}(\vec{v}_1) = \vec{I}^{(4)}(\vec{v}_0) + \int_0^1 \mathrm{d}t \, \left(\frac{\partial A}{\partial t} \cdot \vec{I}^{(3)}(\vec{v}_0) + (A(1) - A(t)) \cdot \frac{\partial A}{\partial t} \cdot \vec{I}^{(2)}(t) \right)$$

Improves evaluation time from
$\mathcal{O}(2$ days$) \to \mathcal{O}(20$ min$)$
@ 20 digits precision

cf. AMFlow
[Liu, Ma '23]

SUMMARY

- We found the complete UT basis and determined the analytic boundary values
- We showed that double-pentagon top sector integrals are related to the integrals known from $\mathcal{N} = 4$ sYM
- We also found the complete alphabet and the symbol space
- Allows for NNLO planar amplitudes computation of $2 \to 4$ massless QCD processes

II. Predicting alphabet letters

Structure of the alphabet

- General form:

$$\mathbb{A} = \left\{ \mathbb{A}_{even}(\vec{v}), \sqrt{Q_i(\vec{v})}, \mathbb{A}_{odd}(\vec{v}) \right\}$$

- Study of singularities of Feynman integrals: **Landau analysis** [Bjorken; Landau; Nakanishi '59]
  - Computes the components of the Landau variety, i.e. the algebraic variety in kinematic space on which singularities may lie
  - Output of the analysis: $\mathbb{A}_{even}, Q_i \in \mathbb{Z}[v_1, \ldots, v_n]$
  - E.g. Two-loop six-point

[Fevola, Mizera, Telen '23,'24]
[Helmer, Papathanasiou, Tellander '24]
[He, Jiang, Liu, Yang '23; Jiang, Liu, Xu, Yang '24]
[Caron-Huot, Correia, Giroux '24]
[Correia, Giroux, Mizera '25]

$$\mathbb{A}_{\text{even}} = \{s_{12}, s_{23}, \ldots, s_{12} - s_{123}, \ldots\}$$ 117 polynomial letters

$$\{\sqrt{Q_i}\} = \{\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \ldots\}$$ 39 square root letters

ODD (ALGEBRAIC) LETTERS

- We know from experience, that there are also "odd" letters:

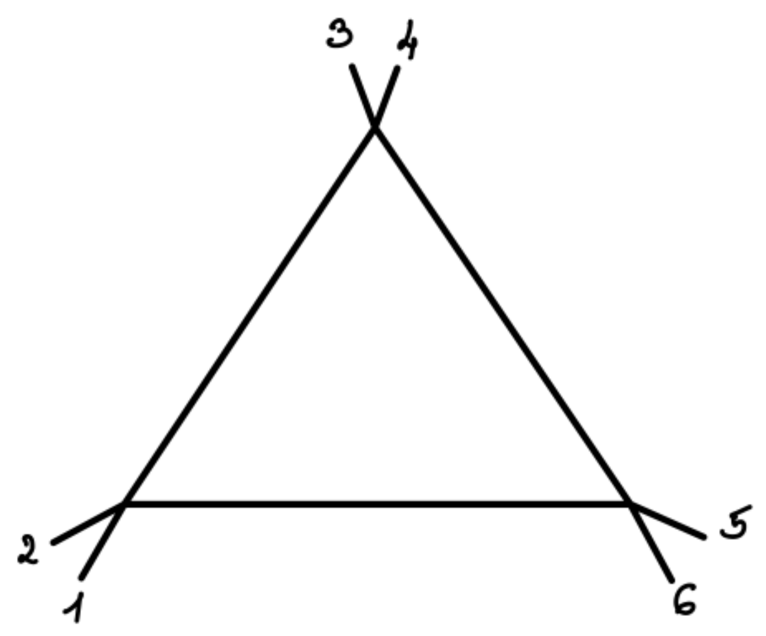
$$W_{odd} = \frac{P(\vec{v}) - \sqrt{Q(\vec{v})}}{P(\vec{v}) + \sqrt{Q(\vec{v})}}$$

$$Q \in \{Q_i\} \cup \{Q_i Q_j\}$$

- Consistent with Landau analysis if [Heller, von Manteuffel, Schabinger '19]

$$\left(P - \sqrt{Q}\right)\left(P + \sqrt{Q}\right) = c \prod_i W_i^{e_i}, \quad W_i \in \mathbb{A}_{even}$$

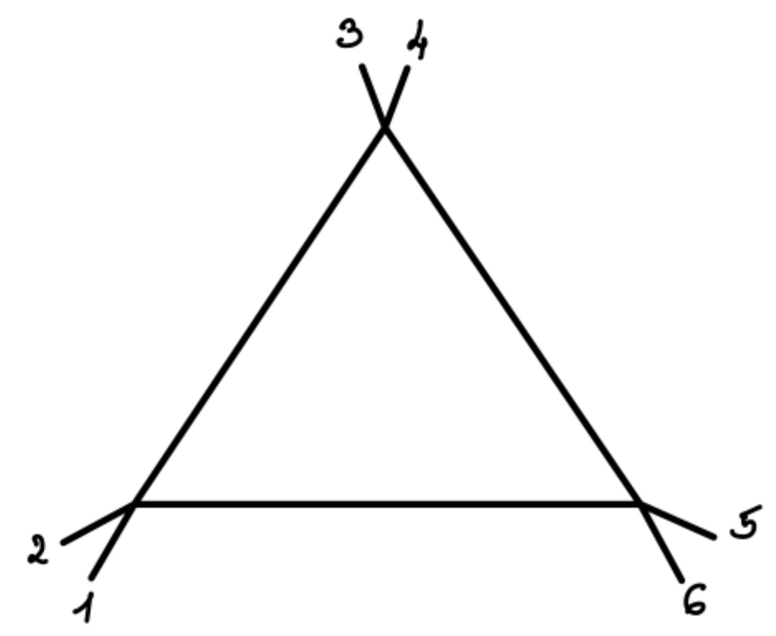
Example



- Consider the odd letters for the three-mass one-loop triangle:

$$\frac{P - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}, \quad \text{where} \quad \lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

EXAMPLE



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- Easy to see that there are three simple ways to turn λ into a perfect square:

$$\lambda(s_{12},s_{34},s_{56})+4s_{12}s_{34}=(s_{12}+s_{34}-s_{56})^2$$

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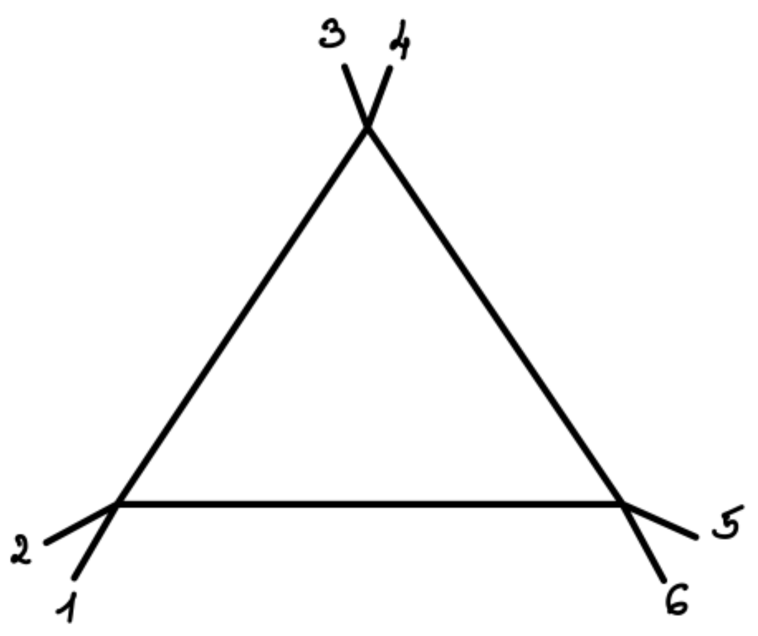

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- However, for embeddings in bigger alphabets, also the following is relevant:

$$\lambda+4(s_{12}s_{56}-s_{12}s_{123}+s_{34}s_{123}-s_{56}s_{123}+s_{123}^2)=(s_{12}-s_{34}+s_{56}-2s_{123})^2$$

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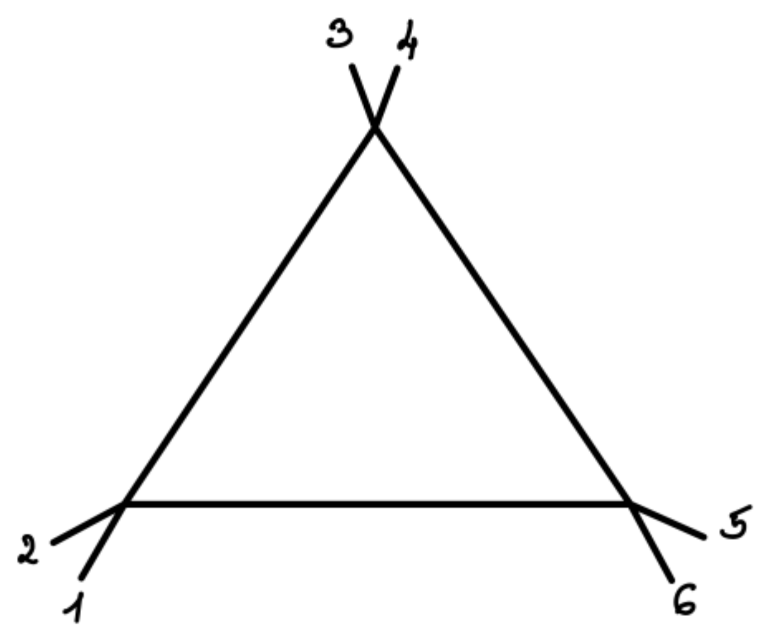

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- Easy to see that there are three simple ways to turn λ into a perfect square:

$$\lambda(s_{12}, s_{34}, s_{56}) + 4s_{12}s_{34} = (s_{12} + s_{34} - s_{56})^2$$

- However, for embeddings in bigger alphabets, also the following is relevant:

$$\lambda + 4(s_{12}s_{56} - s_{12}s_{123} + s_{34}s_{123} - s_{56}s_{123} + s_{123}^2) = (s_{12} - s_{34} + s_{56} - 2s_{123})^2$$

How do we find all possible solutions?

Two approaches

(1) Ansatz for the polynomial P

$$P(\vec{v})^2 = Q(\vec{v}) + c\prod_i W_i^{e_i}$$

- e.g. for $\sqrt{Q} = r_{27}$, i.e. a degree 3 ansatz, assume P has coefficients in $\{-2,\ldots,2\}$:

$$\#P(\vec{v}) \approx 5^{\binom{9+3-1}{3}} \approx 10^{115}$$

(2) Ansatz for the product

$$P(\vec{v})^2 = Q(\vec{v}) + c\prod_i W_i^{e_i}$$

$$\#\prod_i W_i^{e_i} = \binom{48}{6} + \binom{48}{4}\binom{51}{1} + \ldots \approx 2 \cdot 10^7$$

EFFORTLESS

To appear soon [AM, Miczajka WIP]

- Assume that $P(\vec{v})^2 = Q(\vec{v}) + W_i(\vec{v}) \cdot (\ldots)$ for some particular letter W_i
- Then, for every $\vec{v}_0 \in \mathbb{Q}^n$ such that $W_i(\vec{v}_0) = 0$, it follows

$$\sqrt{Q(\vec{v}_0)} \in \mathbb{Q}$$

- Hence, we can filter through the even letters to find a reduced set for any given Q
- E.g.

$$s_{12}s_{56} - s_{12}s_{123} + s_{34}s_{123} - s_{56}s_{123} + s_{123}^2 = 0$$

$$\sqrt{s_{12}^2 + s_{34}^2 + s_{56}^2 - 2s_{12}s_{34} - 2s_{12}s_{56} - 2s_{34}s_{56}} = \sqrt{\frac{(s_{12}s_{56} - s_{123}^2)^2}{s_{123}^2}} \in \mathbb{Q}$$

Effortless

Algorithm 1 The algorithm for construction of algebraic letters

Input: Square root $\sqrt{Q_i}$, list of even letters $\mathbb{A}_{even}$, value of coefficient c

Output: List of algebraic letters involving a square root $\sqrt{Q_i}$

for all $W_j \in \mathbb{A}_{even}$ **do**
    if $\sqrt{Q_i}|_{W_j=0} \in \mathbb{Q}$ **then**
        add W_j to the allowed letters list $\mathbb{A}_{allowed}$
    end if
end for
construct all products $R_k = \prod_{W_j \in \mathbb{A}_{allowed}} W_j^{e_j}$ of degree q
for all R_k **do**
    if $\sqrt{Q + cR_k}$ is perfect square **then**
        $P_k = \sqrt{Q + cR_k}$
    end if
end for
return list of $\dfrac{P_k - \sqrt{Q_i}}{P_k - \sqrt{Q_i}}$

Avilable @

https://github.com/antonela-matijasic/Effortless.git

TWO-LOOP SIX-POINT ALPHABET

- Started with 117 even letters from subsectors and maximal cut
- Using the algorithm, we construct a total of 133 odd letters
  - out of these, **118** of them have a single square root, **15** have two square roots
  - **100** odd letters can be matched to known data from two-loop five-point and one-loop six-point alphabets; **33** letters are completely new

Together with the 39 square roots, we find an alphabet with 117+39+133=**289** letters.

DISCUSSION & OUTLOOK

- For a single square root, it takes $\mathcal{O}(s)$ to get all odd letters
- Incorporated into SOFIA (by Correia, Giroux, Mizera)
- Knowing the alphabet is helpful for efficiently getting the DE & can be used for bottstrap methods
- A more general ansatz for nested square roots?

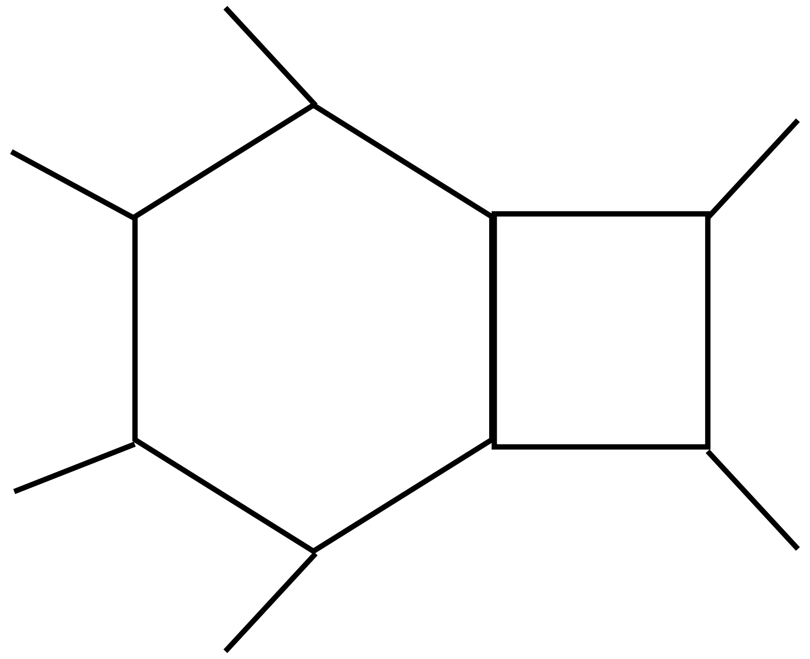
$$\sqrt{Q_1 + Q_2\sqrt{Q_3}}$$

Thank you!

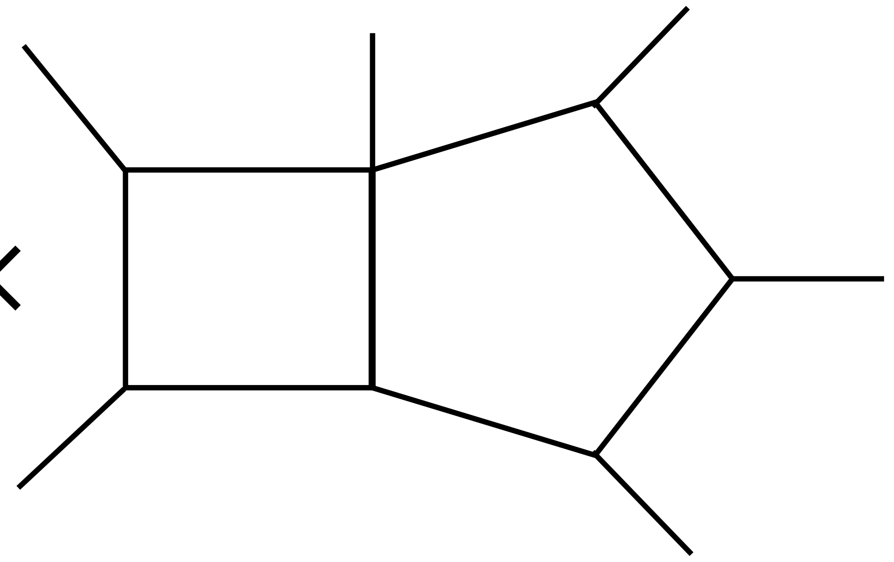
BACK-UP SLIDES

UT INTEGRALS FOR HEXABOX & PENTABOX

$$I_{hb} = \epsilon^4 N_{hb} \times$$



$$I_{pb} = \epsilon^4 N_{pb,i} \times$$



Numerators:

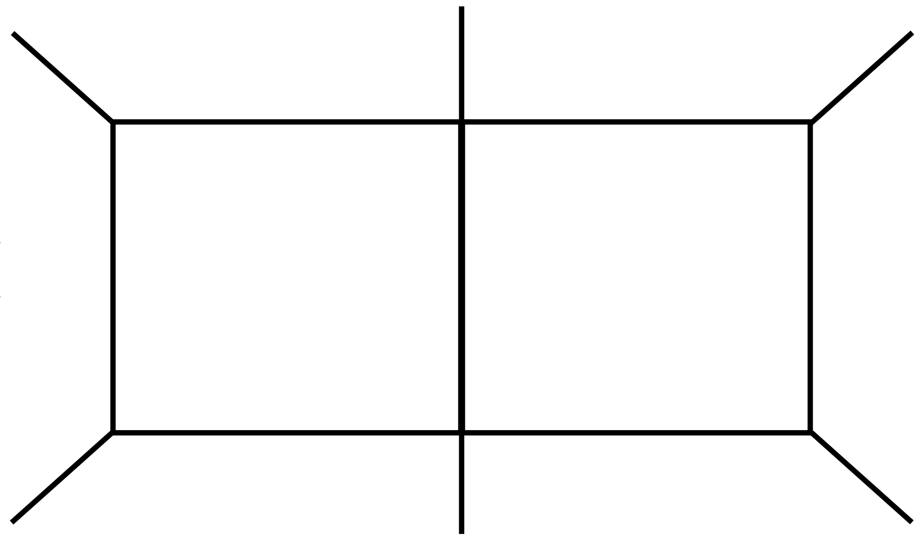
$$N_{hb} = s_{56} \frac{G(l_1, p_1, p_2, p_3, p_4)}{\epsilon_{1234}} (l_1 + p_6)^2$$

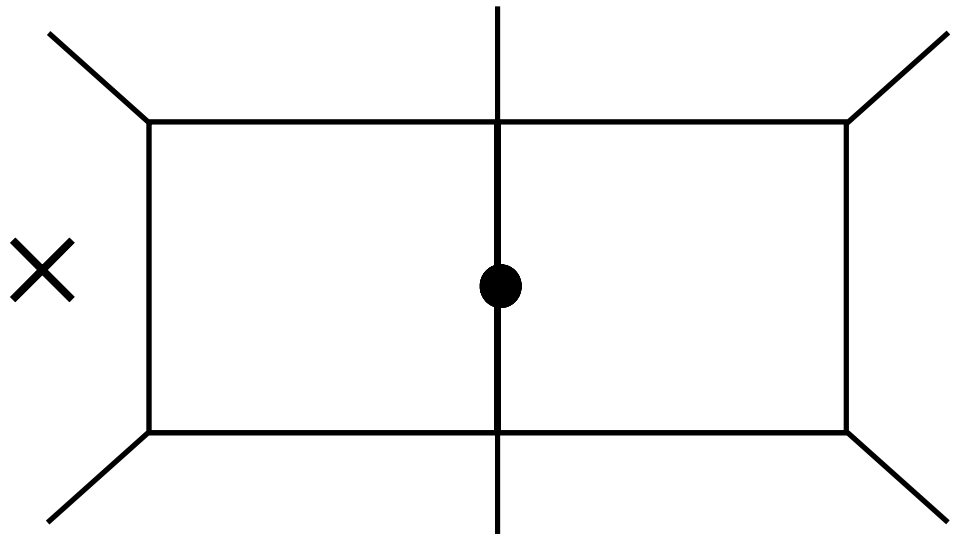
$$N_{pb,1} = s_{56} \frac{G(l_1, p_1, p_2, p_3, p_6)}{\epsilon_{6123}}$$

$$N_{pb,2} = s_{12} s_{23} s_{56} (l_1 + p_6)^2$$

$$N_{pb,3} = 4 \frac{-s_{12}\epsilon_{4561} + s_{123}\epsilon_{5612}}{G(p_1, p_2, p_3, p_6)} G \begin{pmatrix} l_1 & p_1 & p_2 & p_3 & p_6 \\ l_2 & p_1 & p_2 & p_3 & p_6 \end{pmatrix}$$

UT INTEGRALS FOR THE DOUBLE BOX FAMILY

$$I_{db,i} = \epsilon^4 N_{db,i} \times$$ 

$$I_{db,7} = \epsilon^4 N_{db,7} \times$$ 

Numerators:

$$N_{db,1} = -s_{12}s_{45}s_{234}$$

$$N_{db,4} = \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix}$$

$$N_{db,2} = -s_{12}s_{45}(l_1 + p_5 + p_6)^2$$

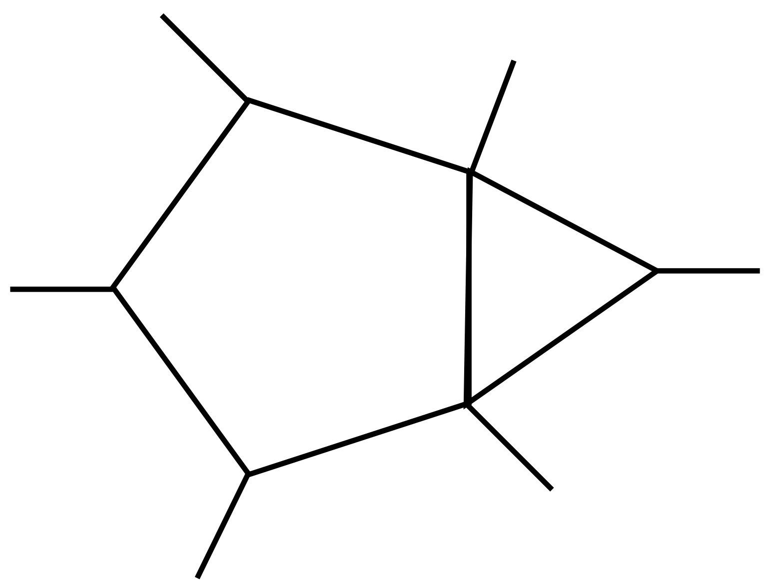
$$N_{db,5} = -\frac{1}{4}\frac{\epsilon_{1245}}{G(p_1,p_2,p_5,p_6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix}$$

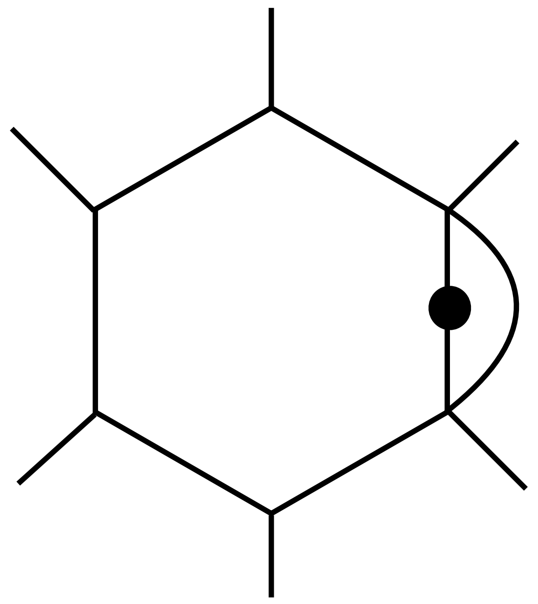
$$N_{db,3} = \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix}$$

$$N_{db,6} = \frac{1}{8}\left[G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{pmatrix} + (l_1 - p_1)^2 (l_2 - p_6 - p_5)^2 (s_{123} + s_{345}) \right]$$

$$N_{db,7} = \frac{\Delta_6}{G(p_1,p_2,p_4,p_5)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}$$

UT INTEGRALS FOR THE PENTATRI & HEXABUB

$I_{pt} = \epsilon^4 N_{pt} \times$ 

$I_{hbb} = \epsilon^3 N_{hbb} \times$ 

Numerators:

$$N_{\text{pt}} = \frac{1}{32\epsilon_{1235}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_3 & p_5 \\ l_1 & p_1 & p_2 & p_3 & p_5 \end{pmatrix}$$

$$N_{\text{hbb}} = \frac{(l_1 + p_6)^2}{32\epsilon_{1234}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_3 & p_4 \\ l_1 & p_1 & p_2 & p_3 & p_4 \end{pmatrix}$$

FIXING THE BOUNDARY VALUES

- Boundary values at $\vec{v}_0 = \{-1, -1, \ldots, -1\}$:

$$I_{\text{pb},2} = -\frac{5}{2} + \frac{13\pi^2}{12}\epsilon^2 + 12\zeta(3)\epsilon^3 + \left(-\frac{89\pi^4}{2160} - \frac{16}{3}\text{Im}\left[\text{Li}_2\left(\rho\right)\right]^2\right)\epsilon^4, \qquad \rho = \frac{1}{2}(1 + i\sqrt{3})$$

$$I_{\text{db},1} = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\text{Im}[\text{Li}_2(\rho)]^2\right)\epsilon^4,$$

$$I_{\text{db},2} = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\ \text{Im}[\text{Li}_2(\rho)]^2\right)\epsilon^4,$$

$$I_{\text{db},6} = -\left(\frac{\pi^4}{540} + \frac{4}{3}\text{Im}[\text{Li}_2(\rho)]^2\right)\epsilon^4,$$

$$I_{\text{hb}} = I_{\text{pb},1} = I_{\text{pb},3} = I_{\text{db},3} = I_{\text{db},4} = I_{\text{db},5} = I_{\text{db},7} = I_{\text{pt}} = I_{\text{hbb}} = 0$$

MOMENTUM TWISTOR PARAMETRIZATION

- We encode the external kinematics via 6 momentum twistors $Z_i = (\lambda_i, \mu_i)$
- We use the SL(4)-transformations to pick a particular parametrization in terms of 8 independent variables x_j

$$Z = \begin{pmatrix} 1 & 0 & x_1 & x_1 x_2 & x_1 x_3 & x_1 x_6 \\ 0 & 1 & 1 & x_8 & 1 & 1 \\ 0 & 0 & 0 & 1 & x_4 & 1 \\ 0 & 0 & 1 & 0 & x_5 & x_7 \end{pmatrix}$$