UNIVERSITY OF LIVERPOOL

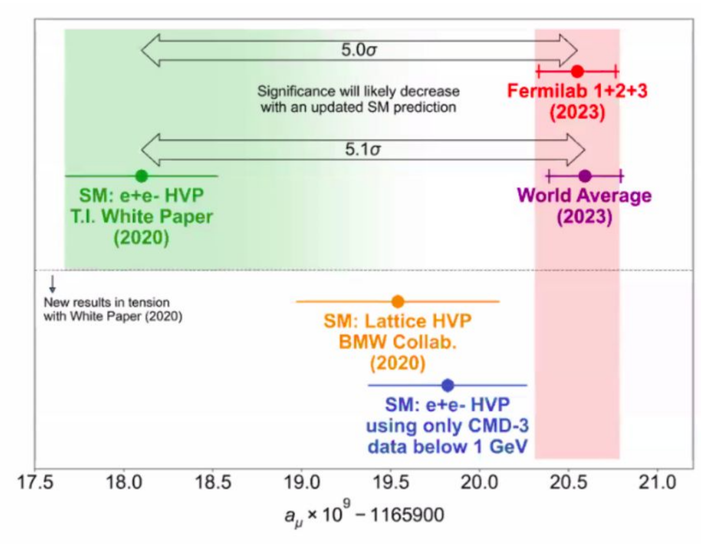
LEVERHULME TRUST

Fast evaluation of Feynman integrals for MC generators

Pau Petit Rosàs

In collaboration with W. Torres Bobadilla

Background & Motivation



To understand the *g-2* anomaly we need [1]

- New data-driven analysis
- MC generators with NNLO $e^+e^- \to x^+x^- + \gamma$
  - $x \in \{\mu, \pi\}$
- Better modelling for pions

➢ Improve the MC Phokhara with
- NNLO
- ϵ, ϵ^2 for NLO
- GVMD for π [2]

[1] R. Aliberti et al (2024)
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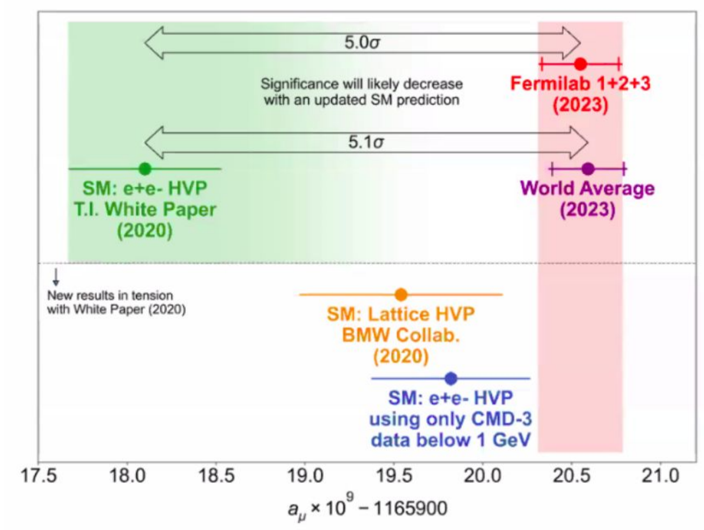

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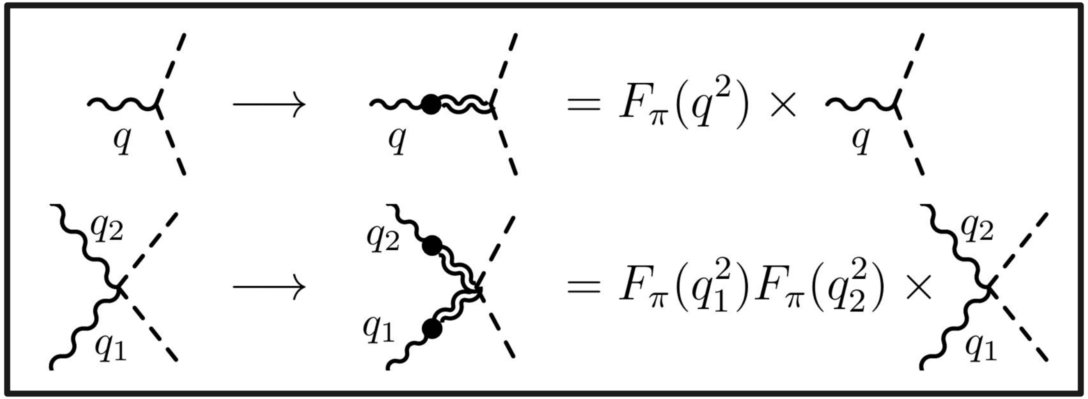
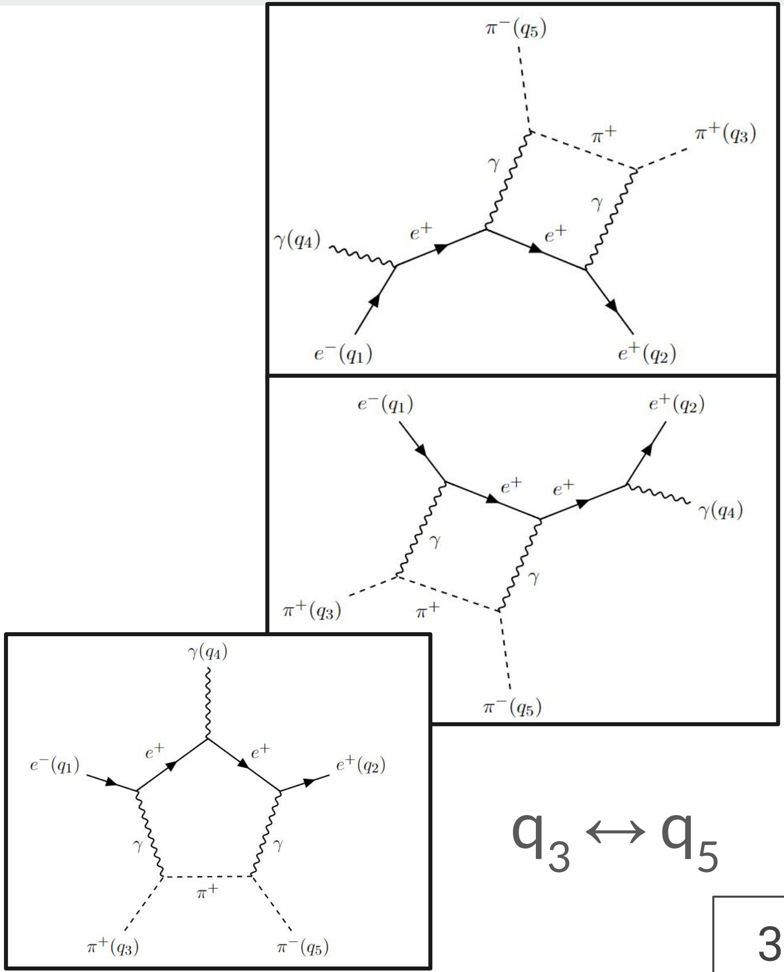
The GVMD model



$$F_\pi(q^2) = \sum_{v=1}^{3} a_v \frac{\Lambda_v}{\Lambda_v - q^2} = \sum_{v=1}^{3} a_v \left(1 + \frac{q^2}{\Lambda_v - q^2}\right)$$

Figure from F. Ignatov, R.N. Lee (2022)

ISR NLO 2γ* diagrams

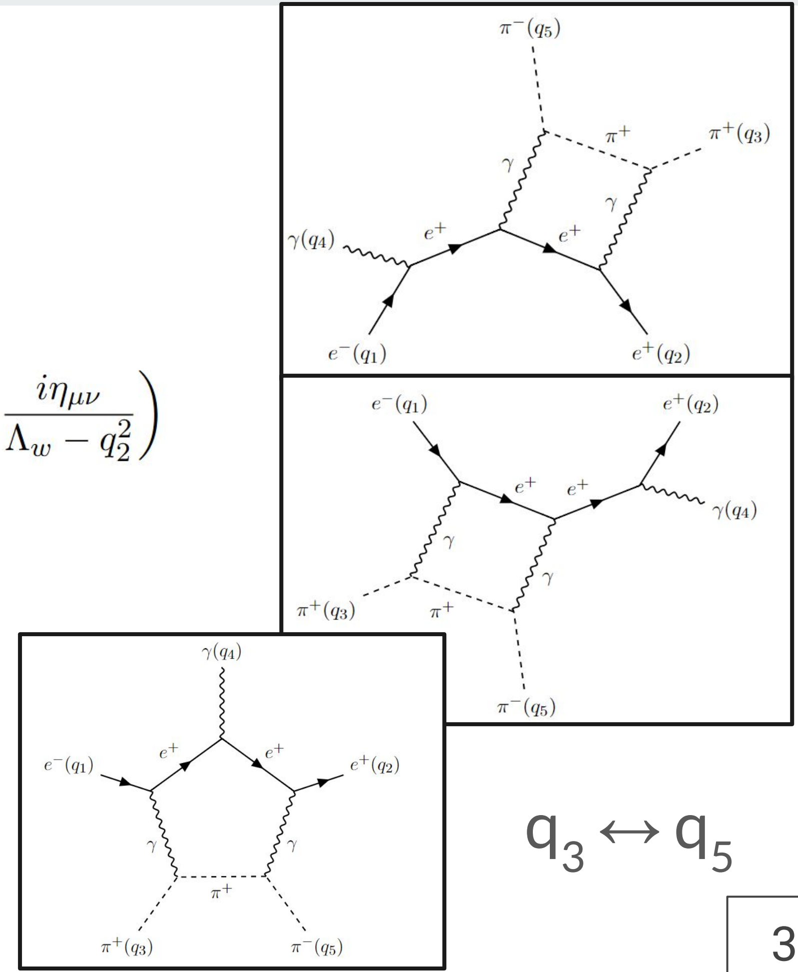


$q_3 \leftrightarrow q_5$

ISR NLO 2ɣ* diagrams

$$A_{TVP} \propto F_\pi(q_1^2) D_{\mu\nu}(q_1^2) F_\pi(q_2^2) D_{\mu\nu}(q_2^2)$$

$$A_{TVP} \propto \sum_{w=1}^{3} \sum_{v=1}^{3} a_v \left(D_{\mu\nu}(q_1^2) + \frac{i\eta_{\mu\nu}}{\Lambda_v - q_1^2} \right) a_w \left(D_{\mu\nu}(q_2^2) + \frac{i\eta_{\mu\nu}}{\Lambda_w - q_2^2} \right)$$



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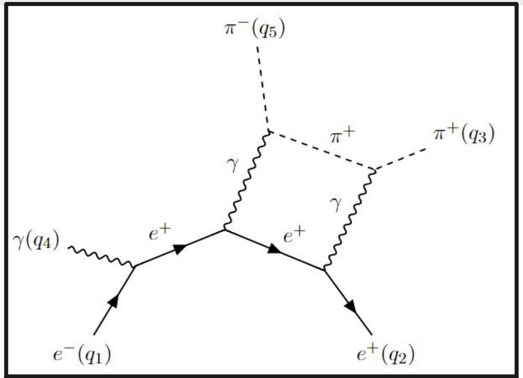
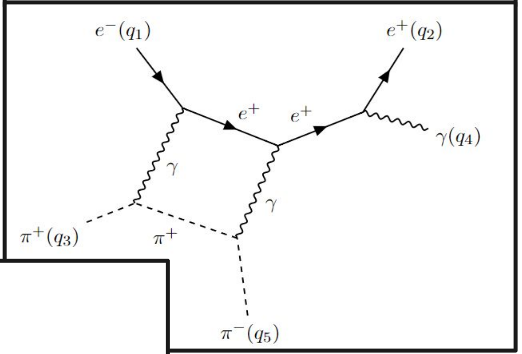
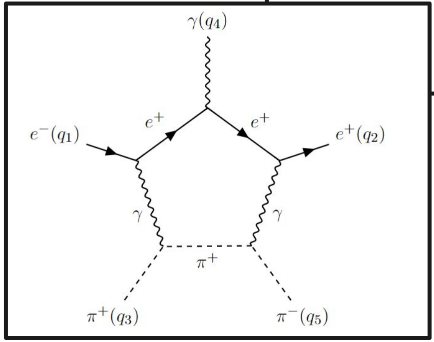
Still a 5 point integral at maximum, but we need to work with up to **9 kinematic variables**. We choose

$$\bar{x} = \{s_{14}, s_{15}, s_{23}, s_{24}, s_{35}, m_e^2, m_\pi^2, m_v^2, m_w^2\}$$

We have 16 possible combinations of m_v, m_w times 2 permutations of the external momenta

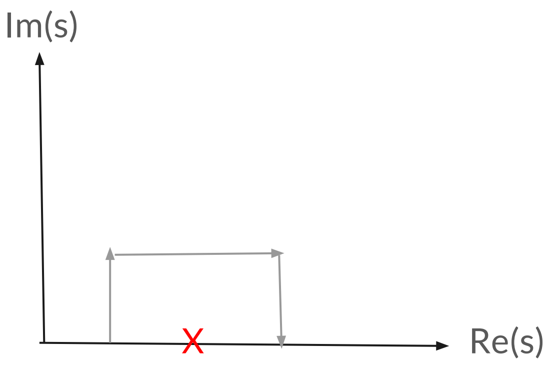






$q_3 \leftrightarrow q_5$

What do we need?



- **Fast integrator** → No mathematica package
- Precise, but no need for 50 significant figures!
- Exploit the fact that we only need to **change values of m_v, m_w**

DiffExp [3] method: write Master Integrals in **differential form**, evolve it variable by variable from a boundary value to the desired final point with the **Frobenius method.** Avoid singularities with analytic continuation.

We could generate a grid of solutions with tools like DiffExp or SeaSyde [4], but dimensionality of the problem is large

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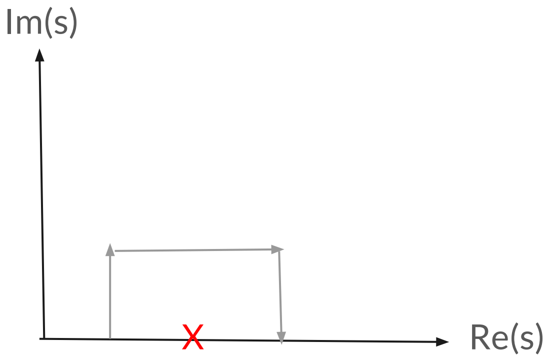

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⇨ What if we evolve the differential equations **numerically**?

C++ integrator

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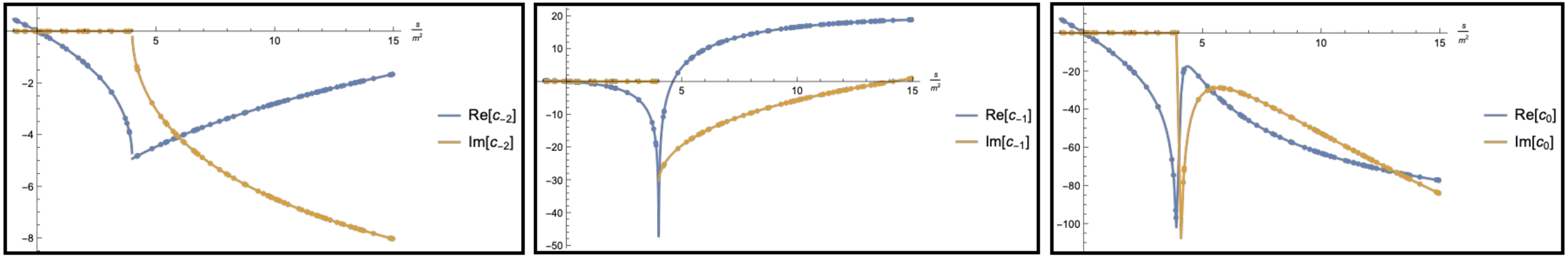
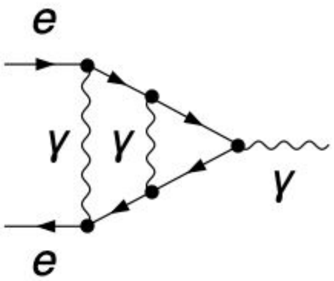
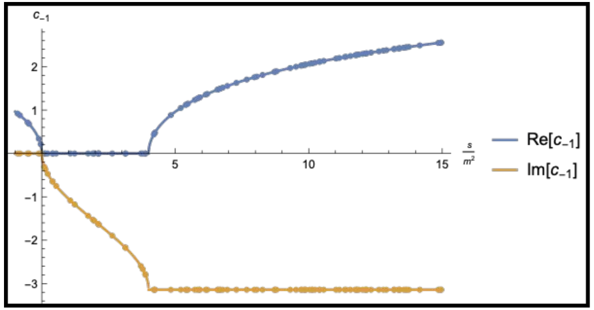
Example

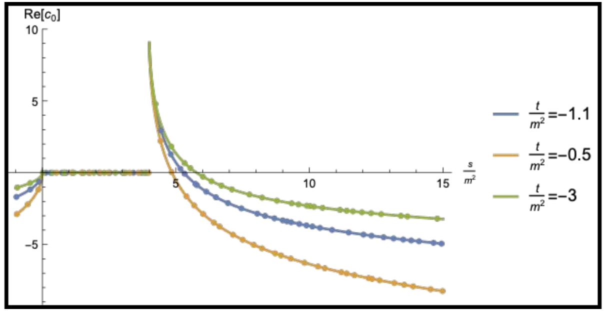


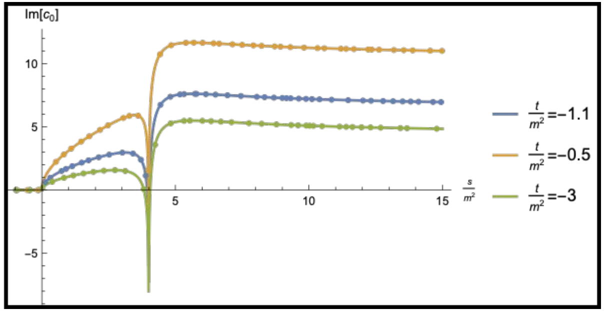


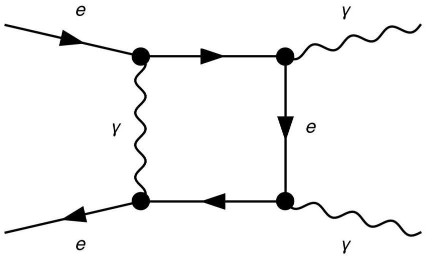
- 10 MIs, 5 orders of ε
- O(μs) per phase-space point
- 7+ significant figures of precision

Example 2.0





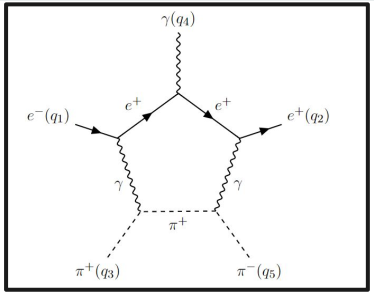




- 8 MIs, 5 orders of ε
- O(*ms*) per phase-space point
- 7+ significant figures of precision

The topologies

$$I^X_{a,b,c,d,e} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{P_1^a P_2^b P_3^c P_4^d P_5^e}$$



	MM	MN	NN
P_1	$m_w^2 - p^2$	$-p^2$	$-p^2$
P_2	$m_\pi^2 - (p - q_3)^2$	$m_\pi^2 - (p - q_3)^2$	$m_\pi^2 - (p - q_3)^2$
P_3	$m_e^2 - (p + q_2)^2$	$m_e^2 - (p + q_2)^2$	$m_e^2 - (p + q_2)^2$
P_4	$m_v^2 - (p + q_{124})^2$	$m_v^2 - (p + q_{124})^2$	$-(p + q_{124})^2$
P_5	$m_e^2 - (p + q_{24})^2$	$m_e^2 - (p + q_{24})^2$	$m_e^2 - (p + q_{24})^2$

29 MIs 25 MIs 21 MIs

Obtaining the DE

- Canonical MIs obtained by studying Leading & Landau singularities in different dimensions
- Use of **FiniteFlow** [5] to reconstruct the DEs with an ansatz based on the **alphabet**

$$d\bar{J} = \epsilon \sum_{i=1}^{n} \mathbf{A}_i dlog(\alpha_i)\bar{J}$$

- Letters of the alphabet predicted by combining **BaikovLetters** [6] and **Effortless** [7]

$$\alpha_i = \frac{P(\bar{x}) + Q(\bar{x})r_k}{P(\bar{x}) - Q(\bar{x})r_k} \qquad \alpha_i = \frac{P(\bar{x}) + Q(\bar{x})r_k r_j}{P(\bar{x}) - Q(\bar{x})r_k r_j}$$

[5] T. Peraro (2019)
[6] X. Jiang (2024)
[7] A. Matijašić, J. Miczajka (xxxx)

A surprise

Reconstruction fails for some elements, but we can reconstruct the DEs without ansatz.

By direct integration of the result, the new letters that appear have the form:

$$\alpha_i = \frac{P(\bar{x}) \pm Q(\bar{x}) r_k + r_j R_{k,\pm}}{P(\bar{x}) \pm Q(\bar{x}) r_k + r_j R_{k,\pm}} \qquad R_{k,\pm} = \sqrt{K(\bar{x}) \pm T(\bar{x}) r_k}$$

First time letters with nested roots in 1-Loop calculations! Already appeared in penta-triangles sectors in High Energy physics calculations of 2-Loops in Refs. [8,9]

- In entries X of **dJ_pent = X*J_triang + …** In particular, $I_{1,0,1,0,1}$, $I_{0,0,1,1,1}$, $I_{0,1,1,0,1}$ for the partial DE of s_{14}
- Also appear for massless photons!
- A lot of open questions…
  - How can we predict them algorithmically? Can we understand the 2 loop structure better from this? Can we rationalize them? What happens if we work with momentum-twistor representation? …

[8] F. Febres Cordero et al (2023)
[9] M. Becchetti et al (2025)

Returning to the integrator...

1. Get partial DE w.r.t. each kinematic variable
2. Input expression for each MI and order of epsilon in terms of letters and other MIs
3. Input values for pre-canonical MIs obtained with AMFlow at a non-singular arbitrary point
4. Input singularities and branch cuts
5. Input expressions for derivatives of letters and square roots
6. Find optimal path between origin and desired final point for each kin. var.
7. Evolve the DE variable by variable in that path:
    a. Multiply the AMFlow values by the canonical factors defined in terms of the current variable
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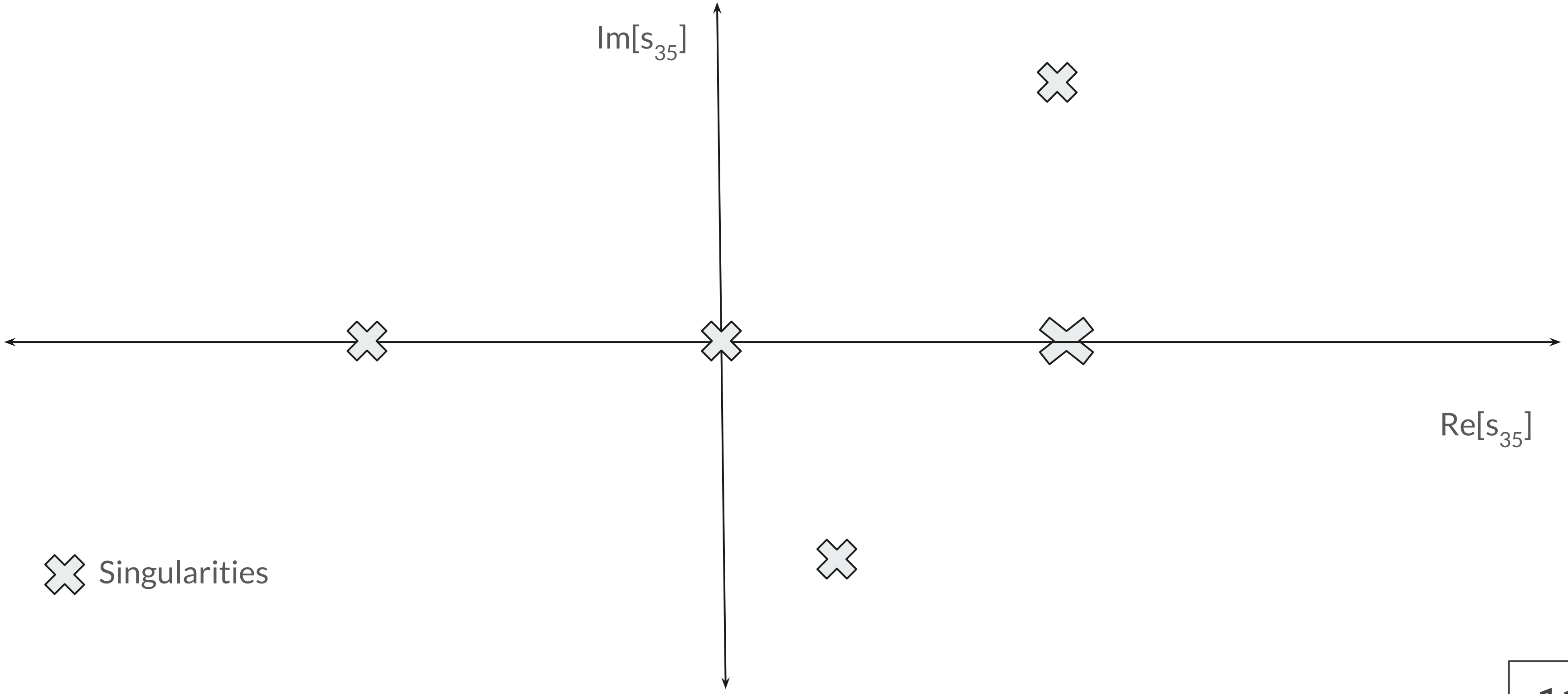
Branch cuts and paths



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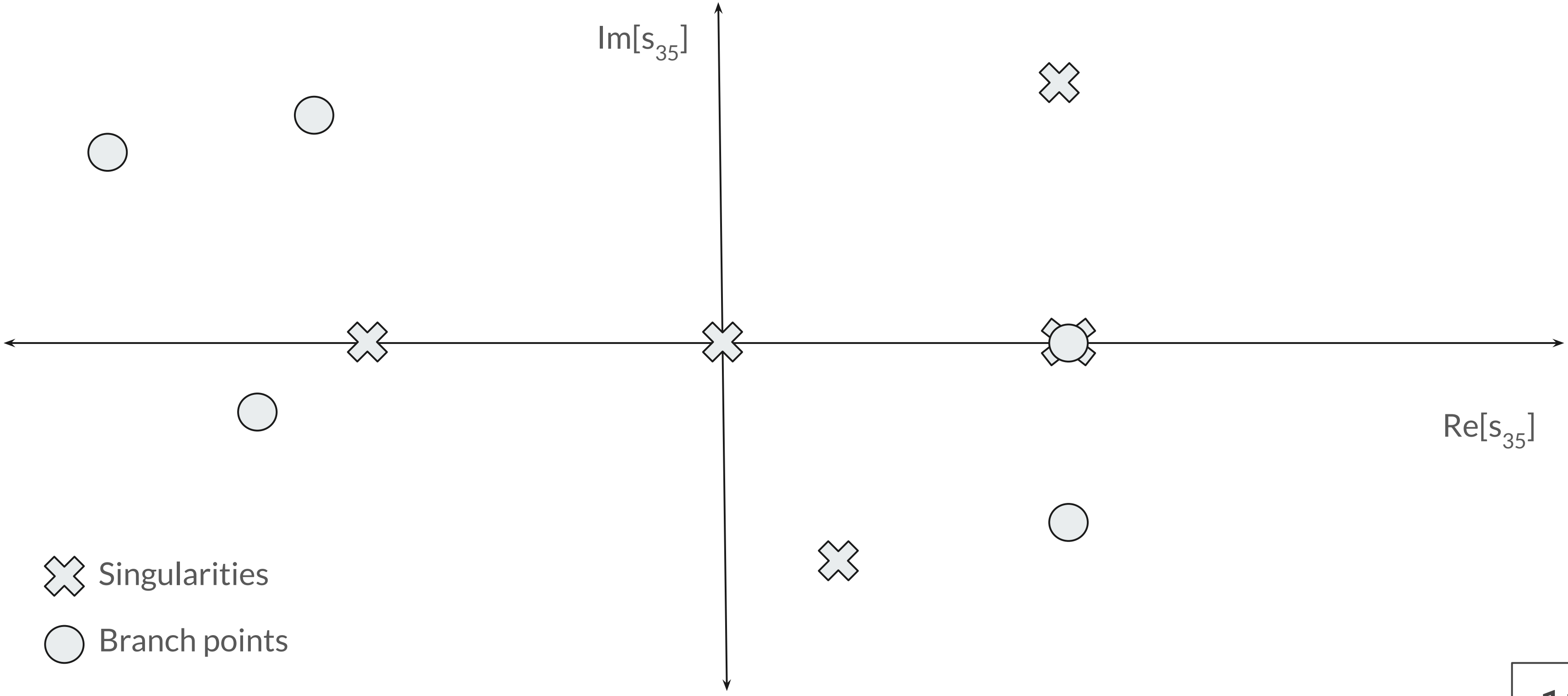

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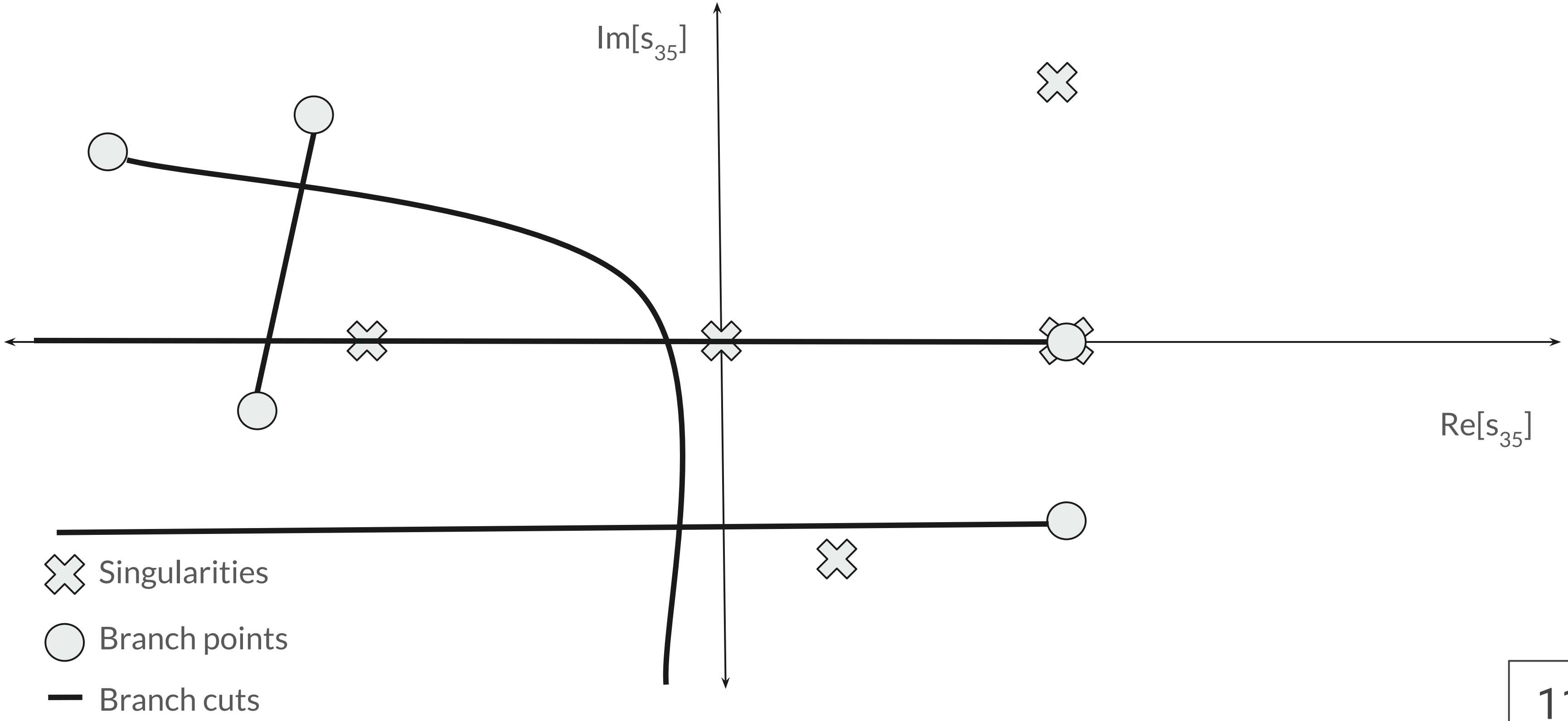

Branch cuts and paths

$$r_i = \sqrt{K_i^n(\bar{x})}$$

$$K^n(\bar{x}) = \Pi_{j=1}^n \left(x - e_j\right)$$

$$\Pi_{j=1}^n \left(x - e_j\right) - \lambda = 0$$

Branch cuts and paths

$$r_i = \sqrt{K_i^n(\bar{x})} \Rightarrow K^n(\bar{x}) = \Pi_{j=1}^n (x - e_j) \Rightarrow \boxed{\Pi_{j=1}^n (x - e_j) - \lambda = 0}$$

$$\Rightarrow r_i = \Pi_{j=1}^n \sqrt{x - e_j} \Rightarrow \boxed{e_j + \lambda_j = 0}$$

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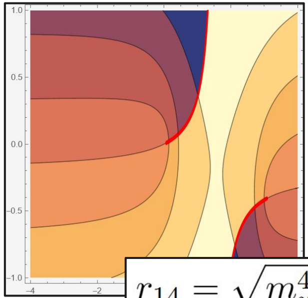
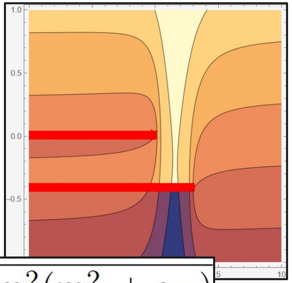
We use the standard convention from mathematical software: branch cuts parallel to the negative real axis.





$$r_{14} = \sqrt{m_v^4 + (m_w^2 - s_{35})^2 - 2m_v^2(m_w^2 + s_{35})}$$

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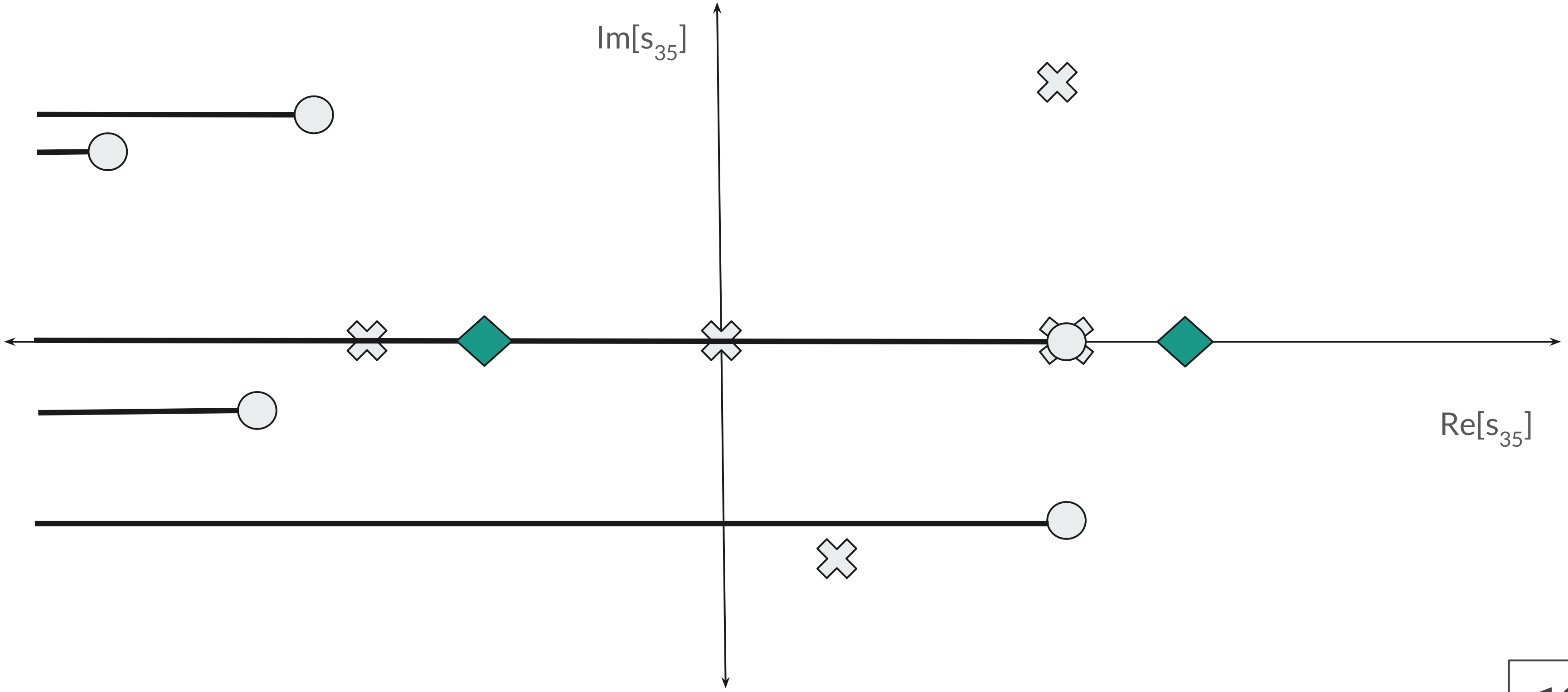

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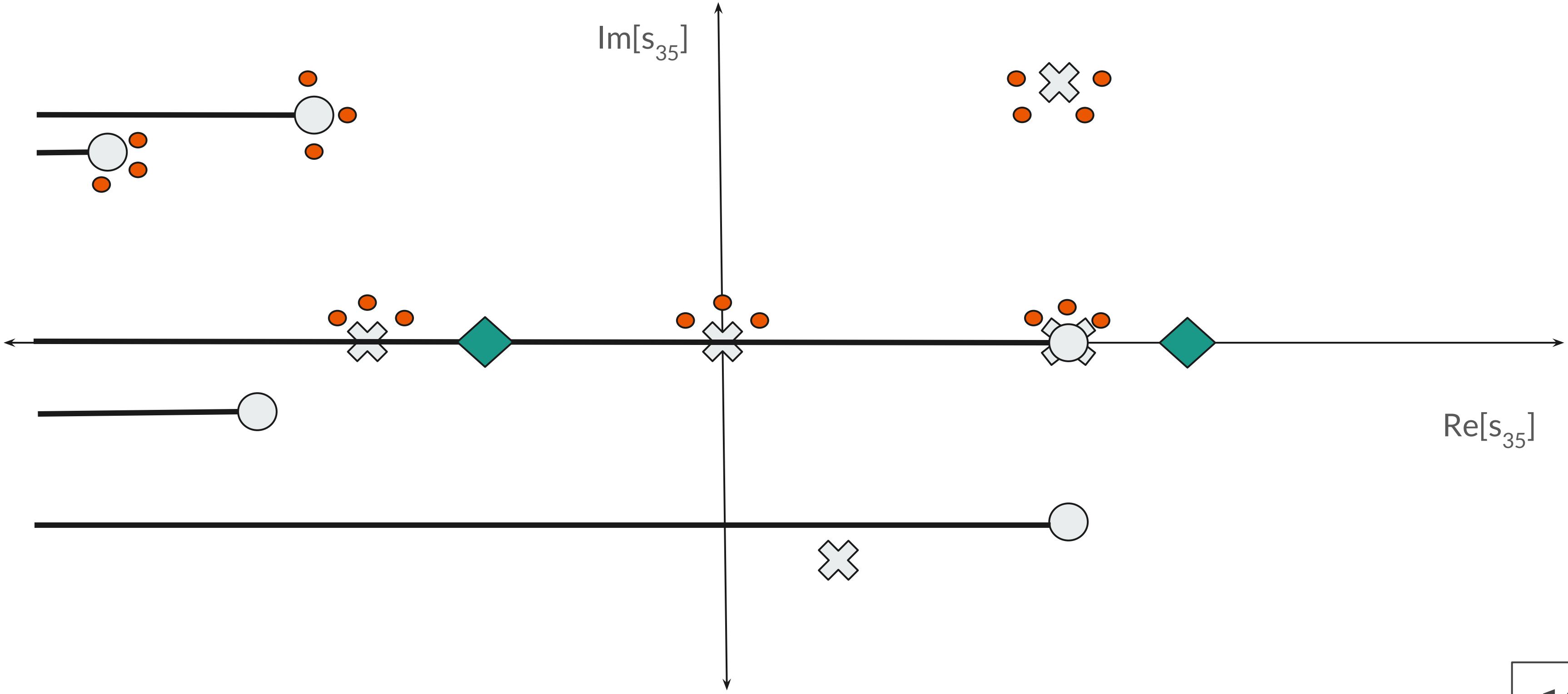

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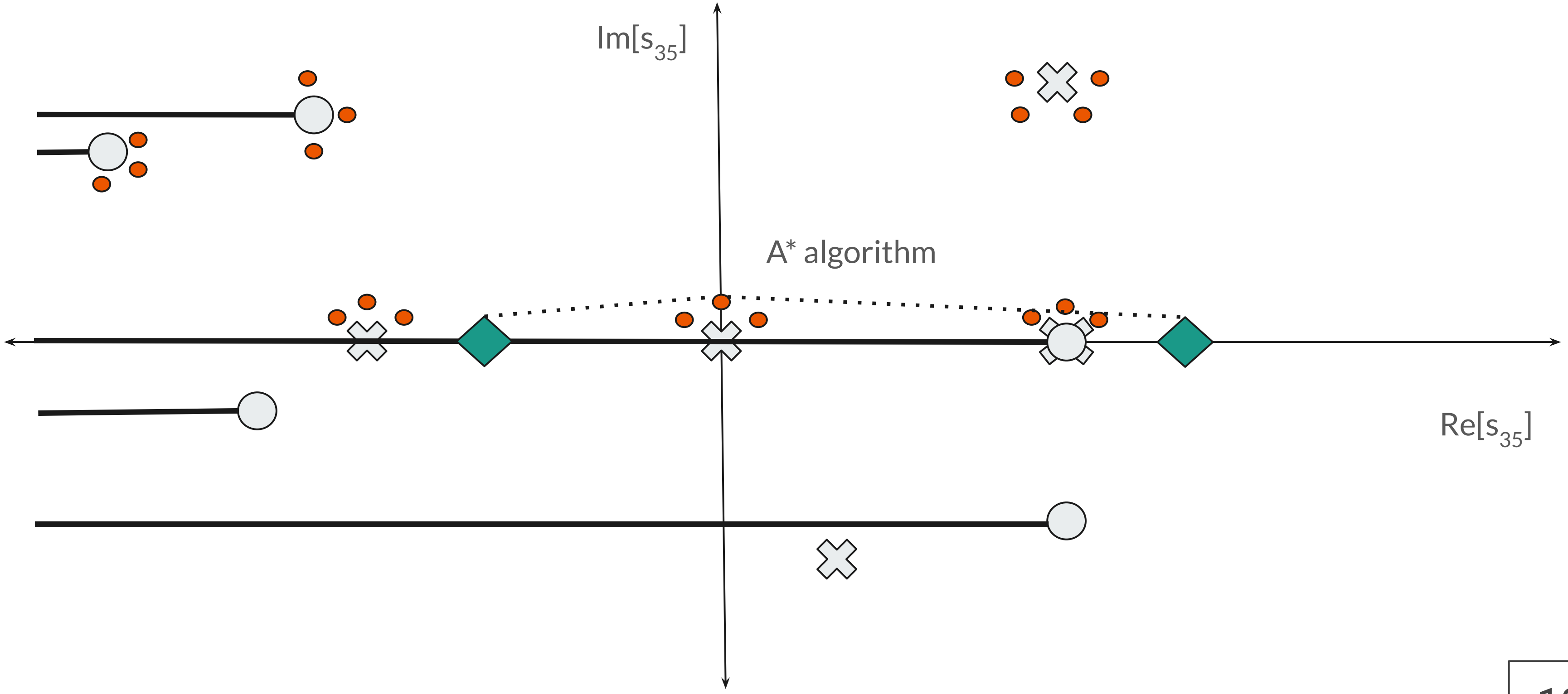

Results

Comparison with Collier. Results of $I_{1,0,1,1,1}$ at finite order

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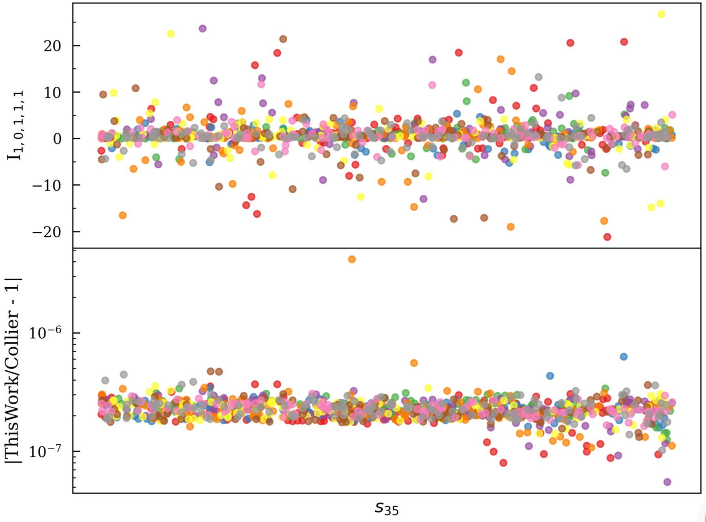

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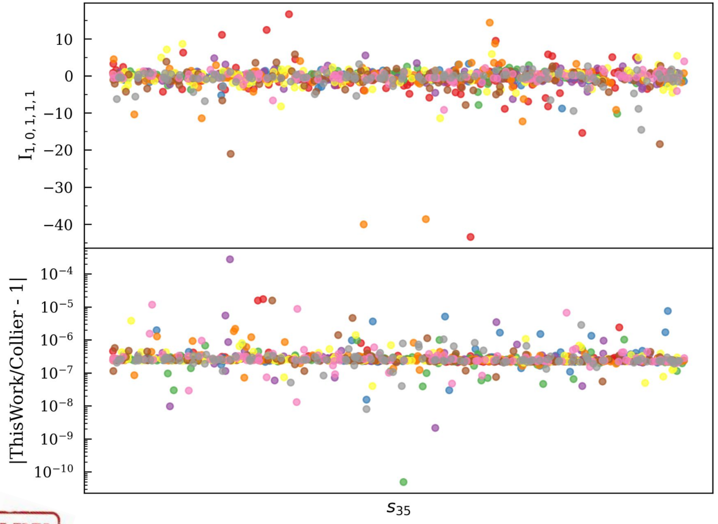

PRELIMINARY

*Thanks to Daniel Gerardo Melo Porras

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Comparison with Collier. Results of $I_{1,0,1,1,0}$ at finite order

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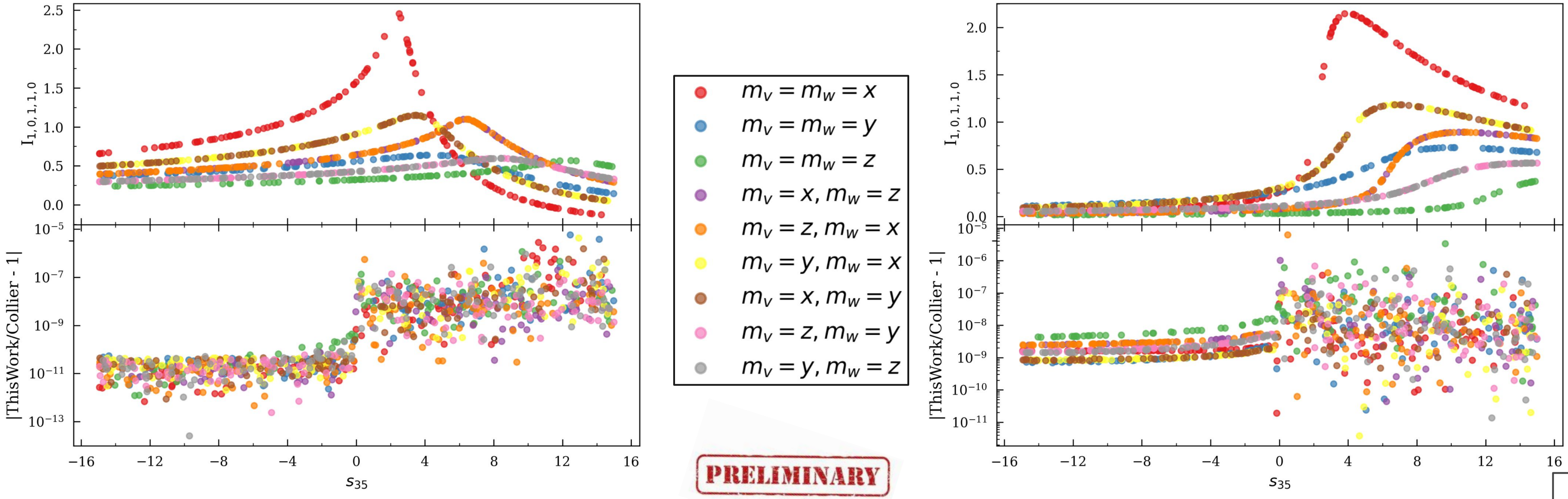

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Summary

1. Interested in $e^+e^- \to \pi^+\pi^- + \gamma$, with massive γ^*
2. Obtained the relevant DEs up to finite order for pentagon topologies
3. Discovery of nested square roots in letters of higher orders
4. Understood the branch cuts and singularities of the DEs
5. Built a C++ integrator capable of calculating integrals with enough precision and speed for MCs
6. Started to verify the results with other tools

Future work

Coming for sure:

- Reconstruct DE for pentagons, more validation
- Net of boundary values
- Publication is in the works
- Arbitrary precision

Even more future work:

- Harder DEs, explore NNLO
- Error estimate of the integrator
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