Modern Integration-by-parts Techniques

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Based on work with

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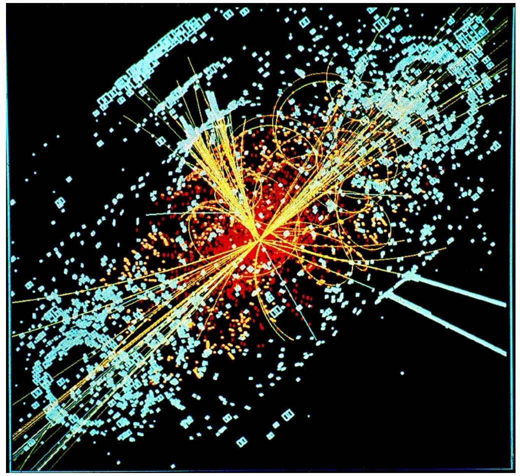
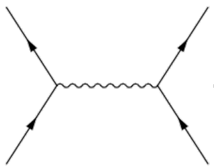
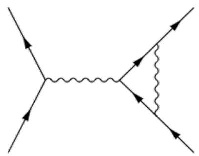
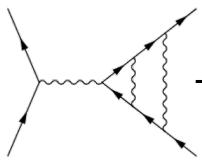
Motivation: Amplitudes

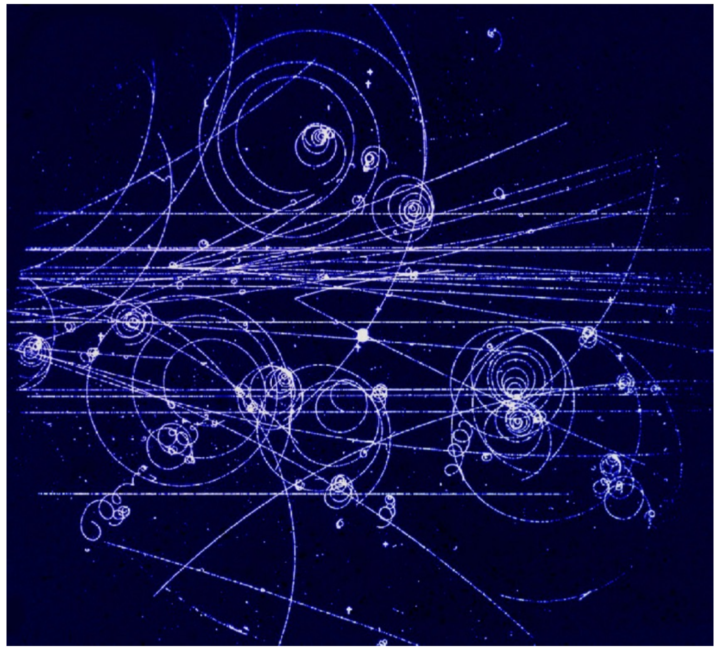


- Observables are given by Amplitudes.
- In perturbative Quantum Field Theory, Amplitudes are given by a sum of Feynman Diagrams

$$\mathcal{A}(e^+e^- \to \mu^+\mu^-) =$$  $+$  $+$  $+ \ldots$

- Each Feynman Diagram corresponds to a Feynman Integral

$$I(n_1, \ldots, n_N) = \int \prod_{a=1}^{L} d^D \ell_a \frac{1}{\rho_1^{n_1} \cdots \rho_N^{n_N}}$$



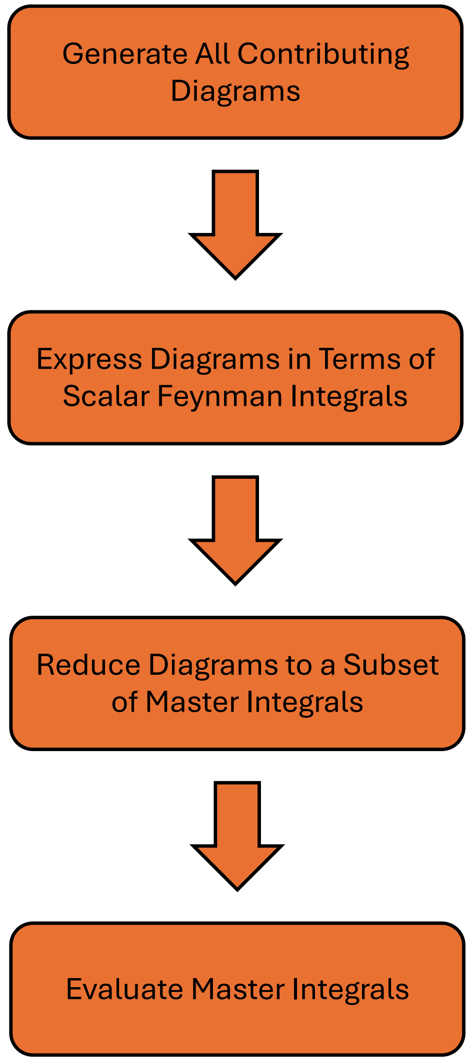
Motivation: Integration-by-parts

- Feynman Integrals $I(n_1, \dots, n_N)$ belong to a Vector Space, the topology defines the space, and the indices n_i define the element. [Smirnov, Petukhov, 2010] [Mastrolia, Mizera, 2019]

- There exists a basis on this vector space, known as the *Master Integrals*.

$$I = \sum_i c_i J_i$$

- Integration-by-parts (IBP) Identities can be used to find the coefficients c_i [Chetyrkin, Tkachov, 1981]



IBPs: Integral Families

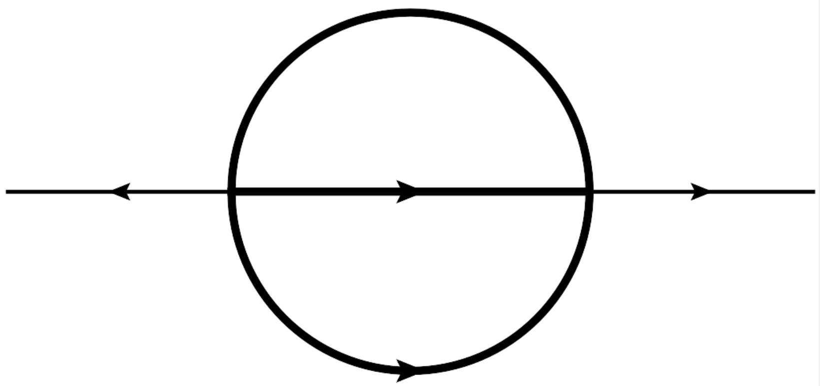
An *integral family/topology* is defined by

- A set of *loop momenta* $\ell_1, \dots, \ell_L$.
- A set of independent *external momenta* $p_1, \dots, p_E$.
- A set of *propagators* $\rho_1, \dots, \rho_N$, of which some can appear as *denominators,* and some as *numerators*.

For Example:

$$\rho_1 = \ell_1^2, \qquad \rho_2 = \ell_2^2, \qquad \rho_3 = (\ell_1 + \ell_2 - p)^2$$

$$\rho_4 = \ell_1 \cdot p, \qquad \rho_5 = \ell_2 \cdot p$$



IBPs: The Laporta Approach

[Laporta, 2000] [LiteRed, Reduze, FIRE, Kira]

- IBP Identities give us relations between integrals in a given family.

- We can input values of the initial vector $\vec{n}$ into these identities to generate a system of equations, this is known as seeding.

- The aim is to row-reduce this matrix until there exists an equation of the form

$$(Target) - \sum_i c_i J_i = 0$$

$$0 = \int \prod_{a=1}^{L} d^D \ell_a \frac{\partial}{\partial \ell_b^\mu} \frac{q_\alpha^\mu}{\rho_1^{n_1} \cdots \rho_N^{n_N}}$$

$$q_\alpha : \{\ell_1, \dots, \ell_L, p_1, \dots, p_E\}$$

$$0 = \sum_i (\alpha_i + \vec{\beta}_i \cdot \vec{n}) I[\vec{n} + \vec{\gamma}_i]$$

$\Downarrow$ *Seeding $\vec{n}$-values*

$$\begin{pmatrix} \# & \cdots & \# \\ \vdots & \ddots & \vdots \\ \# & \cdots & \# \end{pmatrix} \begin{pmatrix} I(\dots) \\ \vdots \\ I(\dots) \end{pmatrix} = 0$$

IBPs: Limitations

Current State-of-the-art:

2-Loop: 5- or 6-point

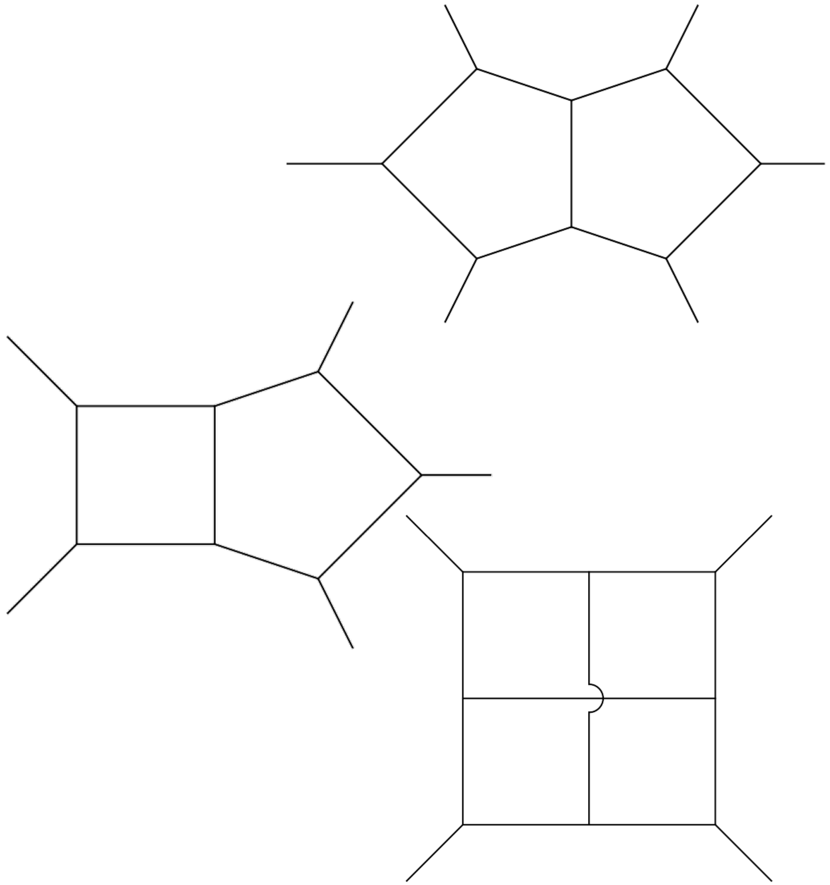
3-Loop: 4-point

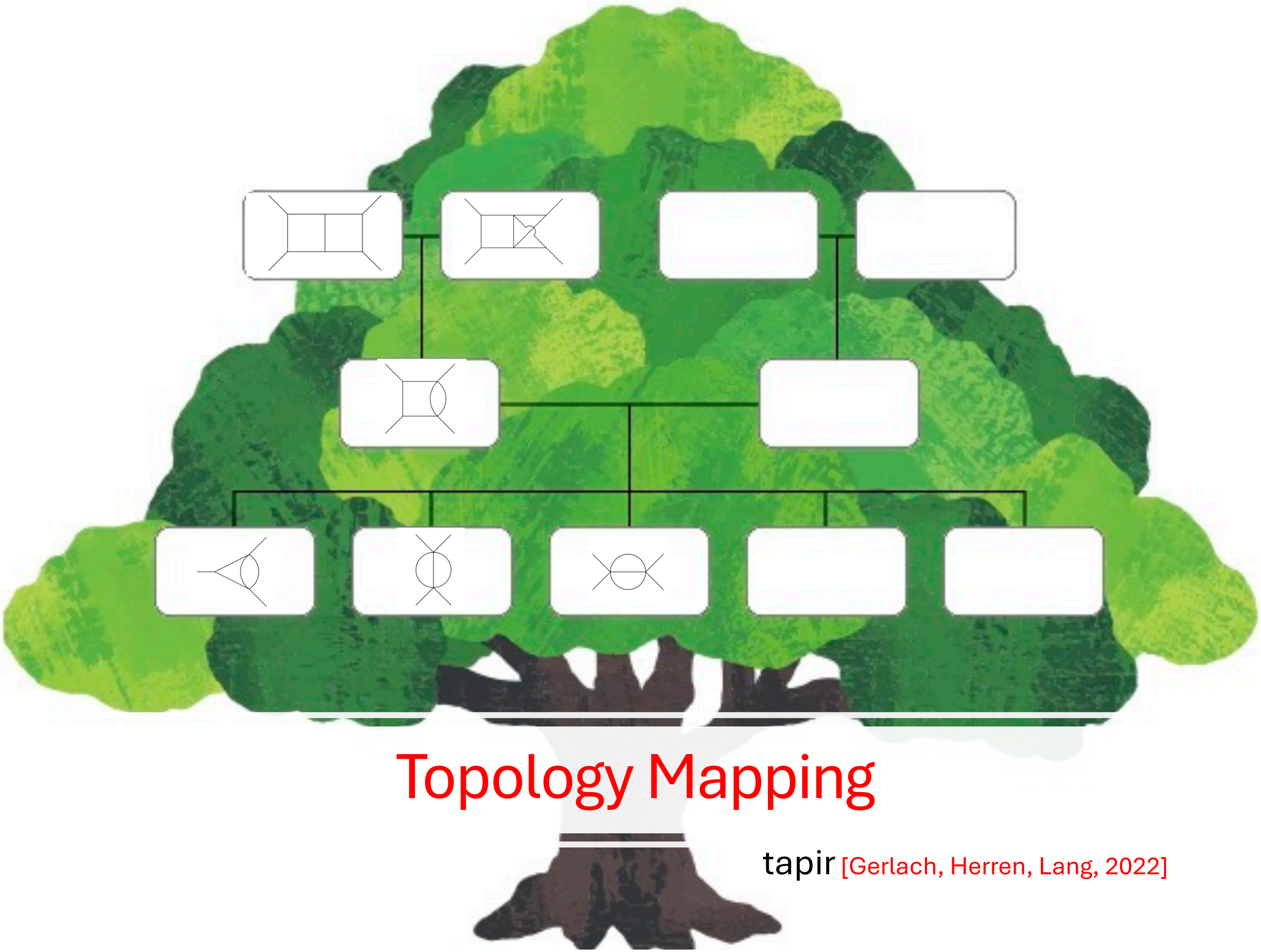
[Abreu, Ita, Moriello, Page, Tschernow, Zeng, 2020]
[Chakraborty, Gambuti, 2022]
[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi, 2023]
[De Laurentis, 2024]
[Bercini, 2024]
[Henn, Matijašić, Miczajka, Peraro, Xu, Zhang, 2024]
[Gehrmann, Henn, Jakubčík, Lim, Mella, 2024]
[Henn, Torres Bobadilla, Lim, 2023/24]

Multi-loop: Increased number of Equations and Variables, larger System of Equations.

Multi-scale: More External Legs and Masses mean more parameters to keep track of when performing row reduction.

High Rank: For Amplitudes calculations, one has to deal with integrals of higher ranks

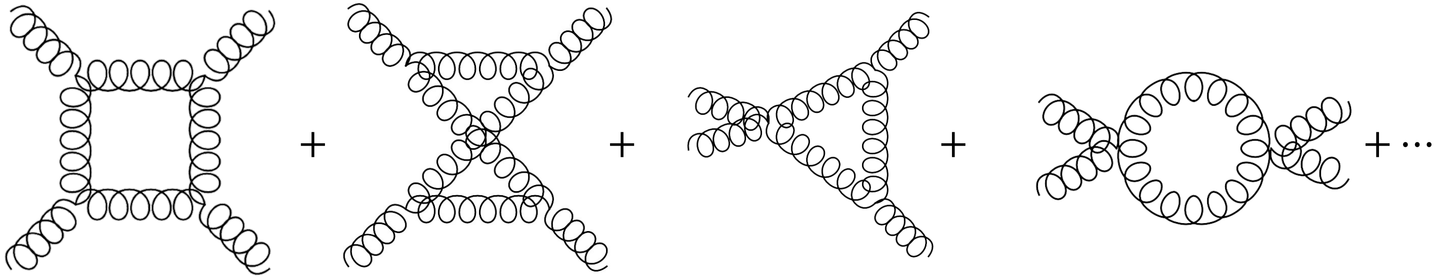


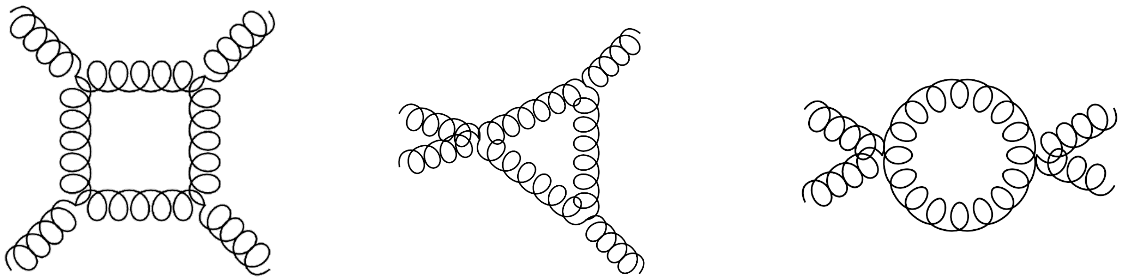


Topology Mapping

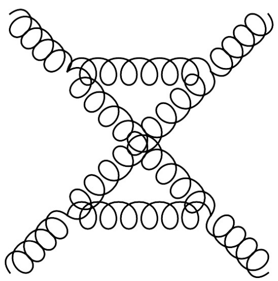
tapir [Gerlach, Herren, Lang, 2022]

$$\mathcal{A}^{L=1}(gg \to gg) =$$





(Permutations)



Topology Mapping: Sub-Topologies

- Loop momenta $\vec{\ell}$, external momenta $\vec{p}$
- Topology A: $\{\vec{\rho}_A(\vec{\ell},\vec{p})\}$
- Topology B: $\{\vec{\rho}_B(\vec{\ell},\vec{p})\}$

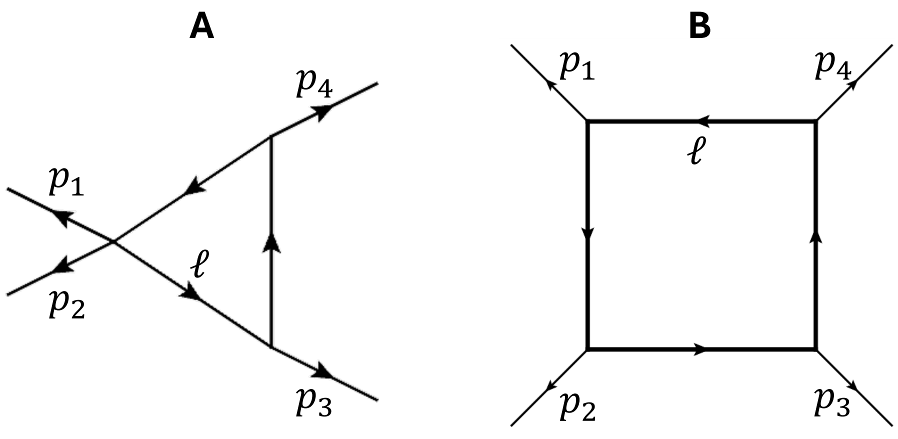
Topology A is a sub-topology of topology B if and only if there exists a *special linear* transformation

$$\vec{\ell} \to \vec{\ell}' = M^{int} \cdot \vec{\ell} + M^{ext} \cdot \vec{p}$$

such that

$$\vec{\rho}_A(\vec{\ell}',\vec{p}) \subseteq \vec{\rho}_B(\vec{\ell},\vec{p})$$

[Dave, Torres Bobadilla, 2024]



$$\vec{\rho}_A = \{(\ell - p_3)^2, (\ell + p_1 + p_2)^2, \ell^2\}$$

$$\vec{\rho}_B = \{\ell^2, (\ell - p_1)^2, (\ell - p_1 - p_2)^2, (\ell - p_1 - p_2 - p_3)^2\}$$

$$\ell \to \ell - p_1 - p_2$$

$$M^{int} = (1), \qquad M^{ext} = (-1 \quad -1 \quad 0)$$

$$\vec{\rho}_A \Big|_{\ell \to \ell - p_1 - p_2} = \{(\ell - p_1 - p_2 - p_3)^2, \ell^2, (\ell - p_1 - p_2)^2\} \subset \vec{\rho}_B$$

Topology Mapping: Symanzik Polynomials

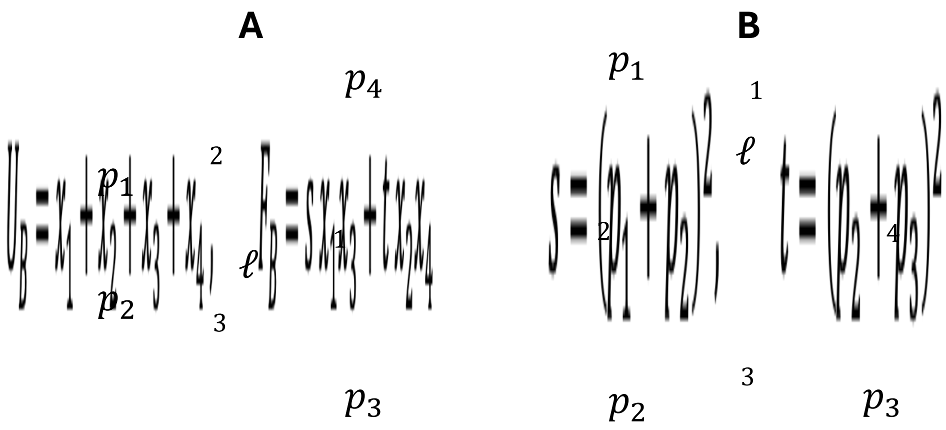
In Feynman Parameters, our Feynman Integral takes the form

$$I(\vec{n}) = \int_0^\infty d^N x\, \delta(1-\sum x_\alpha) \prod_{\alpha=1}^{N} x_\alpha^{n_\alpha - 1} \frac{F^{\frac{LD}{2}-n}}{U^{(L+1)\frac{D}{2}-n}}$$

F and U are *Symanzik, graph-theoretic* polynomials, which are independent of loop momenta shifts.

In this framework a topology is defined by $\{N, U, F\}$.

[Lee, Pomeransky, 2013]



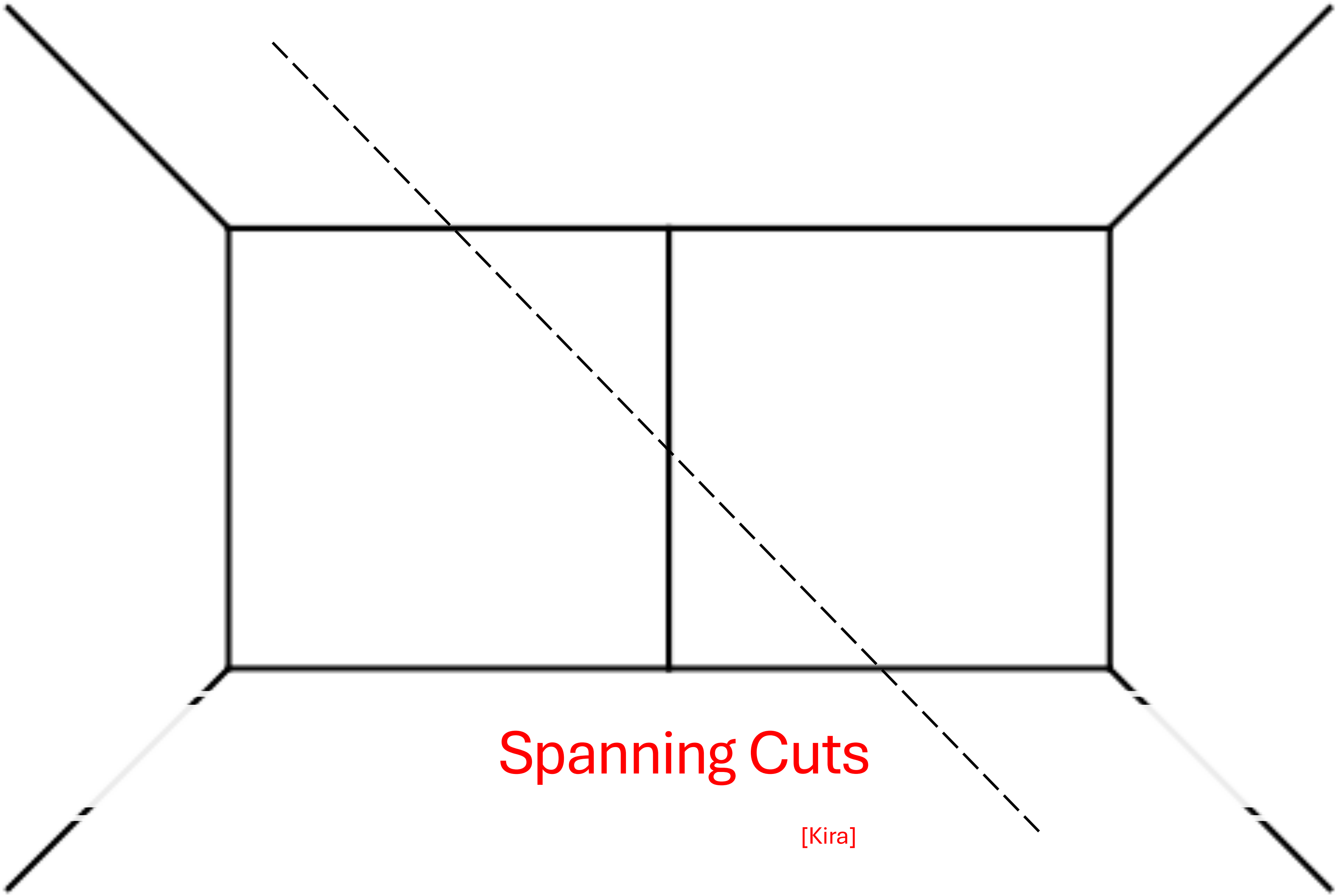
$$U_A = x_1 + x_2 + x_3\,, \qquad F_A = s x_2 x_3$$

$$U_B = x_1 + x_2 + x_3 + x_4\,, \qquad F_B = s\, x_1 x_3 + t\, x_2 x_4$$

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2$$

$$U_A\Big|_{x_1 \leftrightarrow x_2} = x_2 + x_1 + x_3\,, \qquad F_A\Big|_{x_1 \leftrightarrow x_2} = s x_1 x_3$$

$$U_B\Big|_{x_2=0, x_4=x_2} = x_1 + x_3 + x_2\,, \qquad F_B\Big|_{x_2=0} = s\, x_1 x_3$$



Spanning Cuts

[Kira]

Spanning Cuts: What is a cut?

Loosely speaking, cutting a propagator means enforcing that this propagator is on-shell

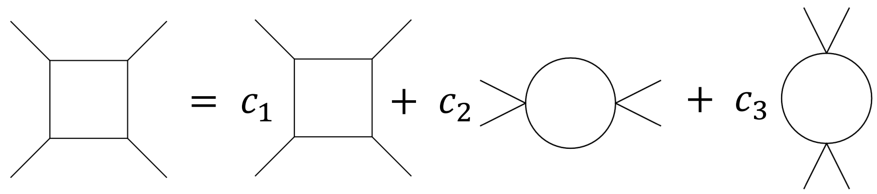
$$\frac{1}{\rho_i} \to \delta(\rho_i)$$

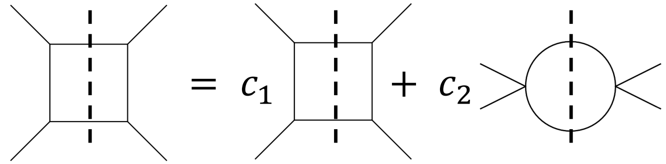
For squared propagators this is more complex, but can be achieved through using IBPs.

If a cut propagator does not appear as a denominator in a Feynman Integral, this integral is *zero on the cut*.

Cuts commute with IBPs.

$$I_C = \sum_i c_i J_{C,i}$$





Spanning Cuts: IBPs with Cuts

We can consider our previous IBP system on a cut C, by making the change

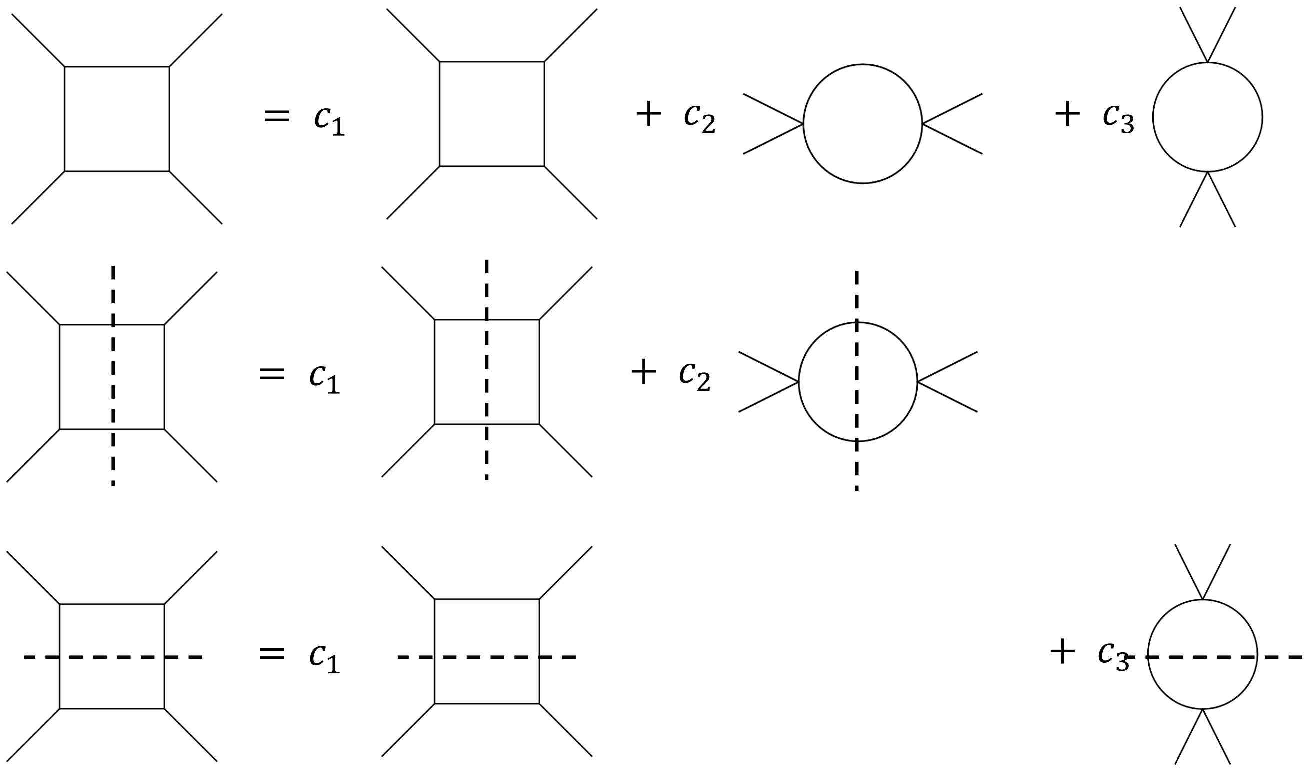
$$I(n_1, \dots, n_N) \to I_C(n_1, \dots, n_N) = \begin{cases} I(n_1, \dots, n_N), & n_i < 0 \; \forall \; i \in C \\ 0, & \text{otherwise} \end{cases}$$

The IBP Equations will therefore become

$$0 = \sum_i (\alpha_i + \vec{\beta}_i \cdot \vec{n}) I[\vec{n} + \vec{\gamma}_i] \Rightarrow 0 = \sum_i (\alpha_i + \vec{\beta}_i \cdot \vec{n}) I_C[\vec{n} + \vec{\gamma}_i]$$

This means we can remove a lot of variables/equations from our system.

Spanning Cuts: Finding a Spanning Set



Syzygies

[Blade, NeatIBP]

Syzygies: Reducing Size of System

When generating IBP identities using the Laporta approach, the derivative acting on the propagators will usually give rise to integrals with a larger value of n_i than the seed integral

$$0 = \int \prod_{a=1}^{L} d^D \ell_a \frac{\partial}{\partial \ell_b^\mu} \frac{q_\alpha^\mu}{\rho_1^{n_1} \cdots \rho_N^{n_N}}, \qquad q_\alpha^\mu \frac{\partial}{\partial \ell_b^\mu} \frac{1}{\rho_i^{n_i}} = \frac{1}{\rho_i^{n_i+1}} q_\alpha^\mu \frac{\partial}{\partial \ell_b^\mu} \rho_i$$

Let's try to make our identity more generic…

Syzygies: Reducing Size of System

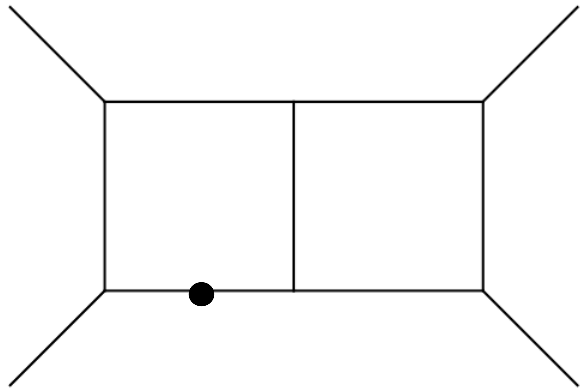
When generating IBP identities using the Laporta approach, the derivative acting on the propagators will usually give rise to integrals with a larger value of n_i than the seed integral

$$0 = \int \prod_{a=1}^{L} d^D \ell_a \frac{\partial}{\partial \ell_b^\mu} \frac{v_b^\mu}{\rho_1^{n_1} \cdots \rho_N^{n_N}}, \qquad v_b^\mu \frac{\partial}{\partial \ell_b^\mu} \frac{1}{\rho_i^{n_i}} = \frac{1}{\rho_i^{n_i+1}} v_b^\mu \frac{\partial}{\partial \ell_b^\mu} \rho_i = \frac{f_i(\rho)}{\rho_i^{n_i}}$$

Let's try to make our identity more generic…

$v_b^\mu \equiv v_b^\mu(\rho)$, we can choose it such that $v_b^\mu \frac{\partial \rho_i}{\partial \ell_b^\mu} = f_i(\rho)\rho_i$

Why?



Syzygies: Where do they come in?

Let's expand $v_b^\mu = P_{b\alpha}(\rho) q_\alpha^\mu$, then the condition becomes

$$P_{b\alpha}(\rho) q_\alpha^\mu \frac{\partial}{\partial \ell_b^\mu} \rho_i = f_i(\rho)\rho_i$$

$$P_{11}(\rho) q_1^\mu \frac{\partial}{\partial \ell_1^\mu} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix} + \cdots + P_{L,L+E}(\rho) q_{L+E}^\mu \frac{\partial}{\partial \ell_L^\mu} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix} - f_1(\rho) \begin{pmatrix} \rho_1 \\ \vdots \\ 0 \end{pmatrix} - \cdots - f_N(\rho) \begin{pmatrix} 0 \\ \vdots \\ \rho_N \end{pmatrix} = 0$$

$$\vec{c}^T M = 0, \qquad M = \begin{pmatrix} q_1^\mu \frac{\partial \rho_1}{\partial \ell_1^\mu} & \cdots & q_1^\mu \frac{\partial \rho_N}{\partial \ell_1^\mu} \\ \vdots & \ddots & \vdots \\ q_{L+E}^\mu \frac{\partial \rho_1}{\partial \ell_L^\mu} & \cdots & q_{L+E}^\mu \frac{\partial \rho_N}{\partial \ell_L^\mu} \\ -\rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\rho_N \end{pmatrix}, \qquad \vec{c} = \begin{pmatrix} P_{11}(\rho) \\ \vdots \\ P_{L.L+E}(\rho) \\ f_1(\rho) \\ \vdots \\ f_{N(\rho)} \end{pmatrix}$$

This is a *Syzygy Equation over a Module* M, we can use *Singular* to find the solutions.

Summary

- Lots of new and exciting advancements being made in the world of IBPs
- By combining all these different ideas, we can push the state-of-the-art further and acquire more precise calculations

Ongoing Questions

- **Syzygy Selection:** Once we have generate syzygy solutions, how do we choose which ones are necessary for an IBP reduction?
- **Seeding Choices:** Once we have a set of IBP Identities, how do we find the minimal amount of seed integrals necessary for a reduction?

Extra Slides

Topology Mapping: Sub-Topologies

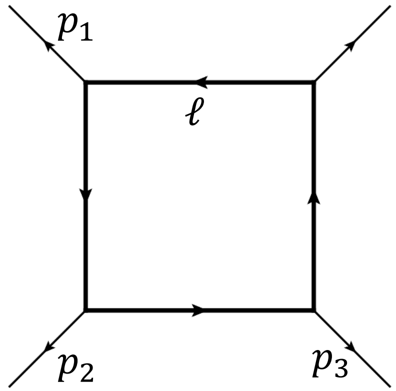
- Topology A: $\{N_A, U_A, F_A\}$
- Topology B: $\{N_B, U_B, F_B\}$

Topology A is a sub-topology of topology B if and only if $N_B \geq N_A$ and there exists some set of $N_B - N_A$ integers R and some permutation matrix P such that

$$U_A(P\vec{x}) = O\left(U_B(\vec{x})\Big|_{x_i \to 0,\, i \in R}\right)$$

$$F_A(P\vec{x}) = O\left(F_B(\vec{x})\Big|_{x_i \to 0,\, i \in R}\right)$$

O is an ordering function for the variables $\vec{x}$



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$$U_B = x_1 + x_2 + x_3 + x_4\,, \qquad F_B = s\, x_1 x_3 + t\, x_2 x_4$$

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2$$

$$R = \{2\} \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U_A(P\vec{x}) = x_2 + x_1 + x_3 = U_B(\vec{x})\Big|_{x_i \to 0,\, i \in R}$$

$$F_A(P\vec{x}) = s x_1 x_3 = F_B(\vec{x})\Big|_{x_i \to 0,\, i \in R}$$

Spanning Cuts: Finding a Spanning Set

A *zero sector* is one where the equation below has a z-independent solution for k_α.

$$\sum_\alpha k_\alpha x_\alpha \frac{\partial}{\partial x_\alpha}(U+F) = U+F$$

Once one has determined all the non-zero sectors $\{S_i\}_{i=1,\ldots,|S|}$, the set of spanning cuts is the minimal set of cuts $\{C_i\}_{i=1,\ldots,|C|}$ such that

$$\forall\, S_i \colon \exists\, C_j | C_j \subseteq S_i$$