# Progress on three-loop four-point integrals with one massive leg

Jungwon Lim

Scattering amplitudes @ Liverpool

**Based on [2410.19088], [2410.22465] and ongoing work with Cesare Carlo Mella and Petr Jakubčík**

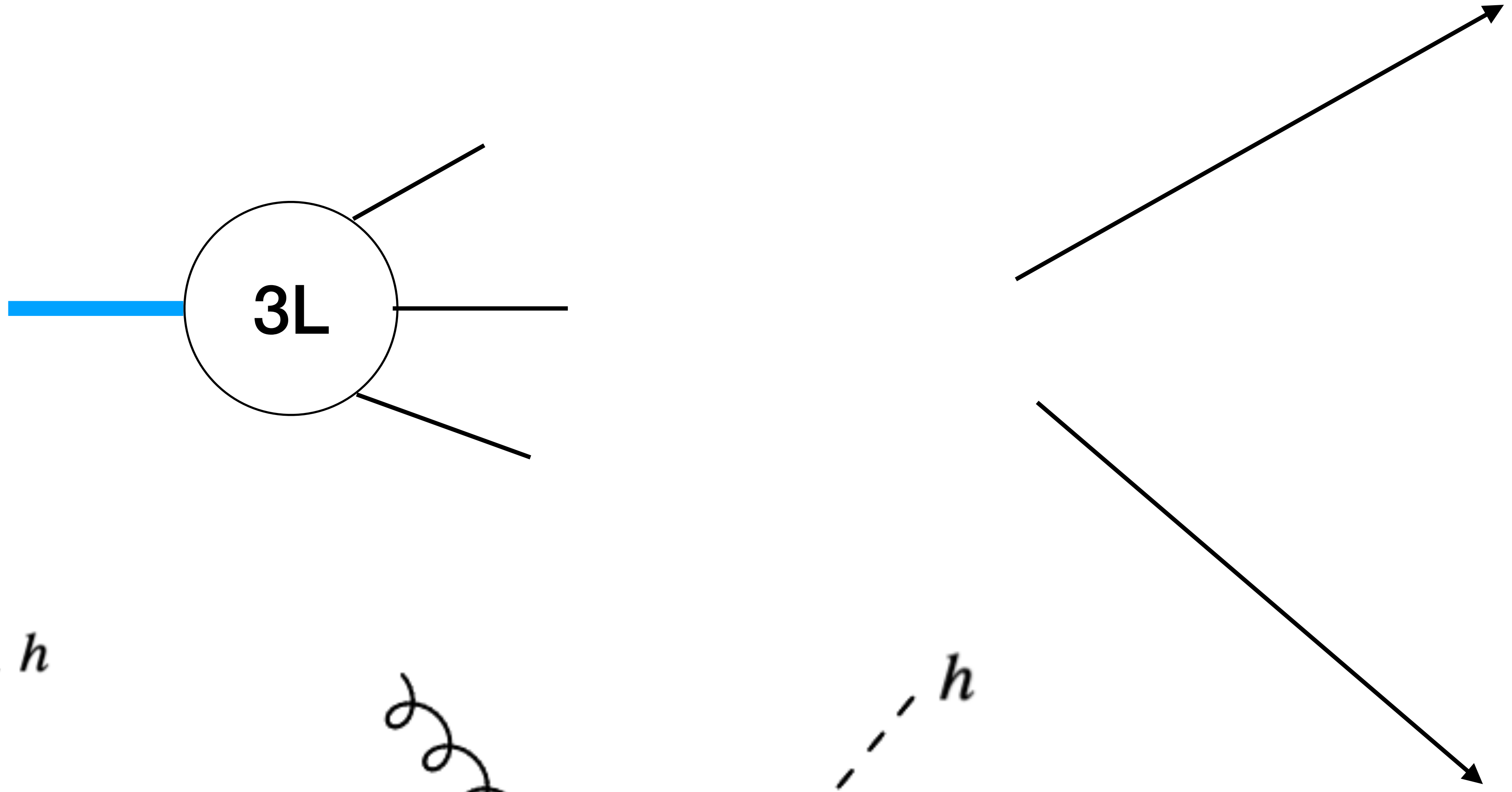
MAX-PLANCK-INSTITUT FÜR PHYSIK

ORIGINS Excellence Cluster

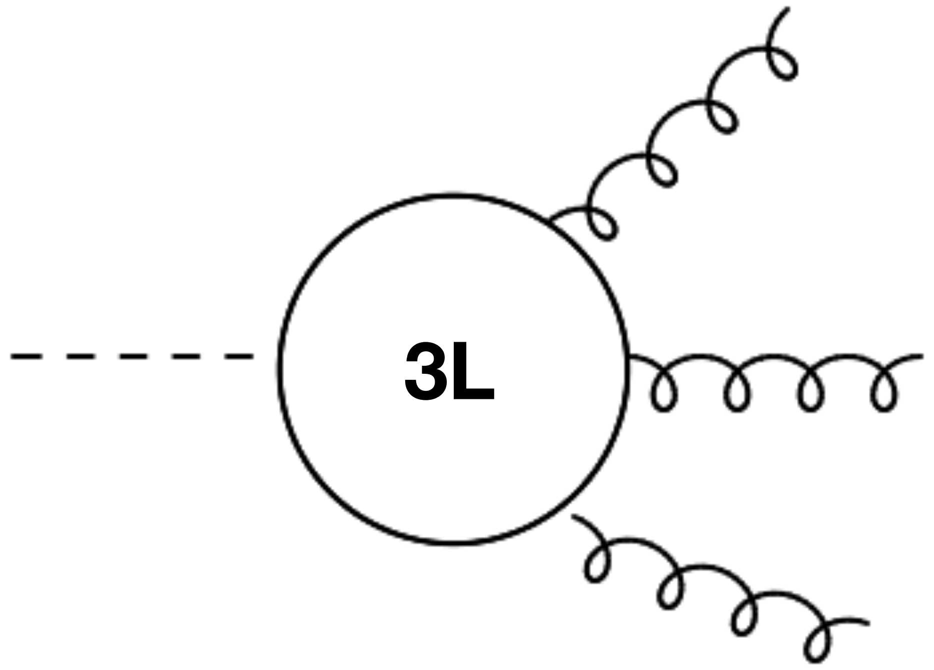
European Research Council
Established by the European Commission

# Motivation

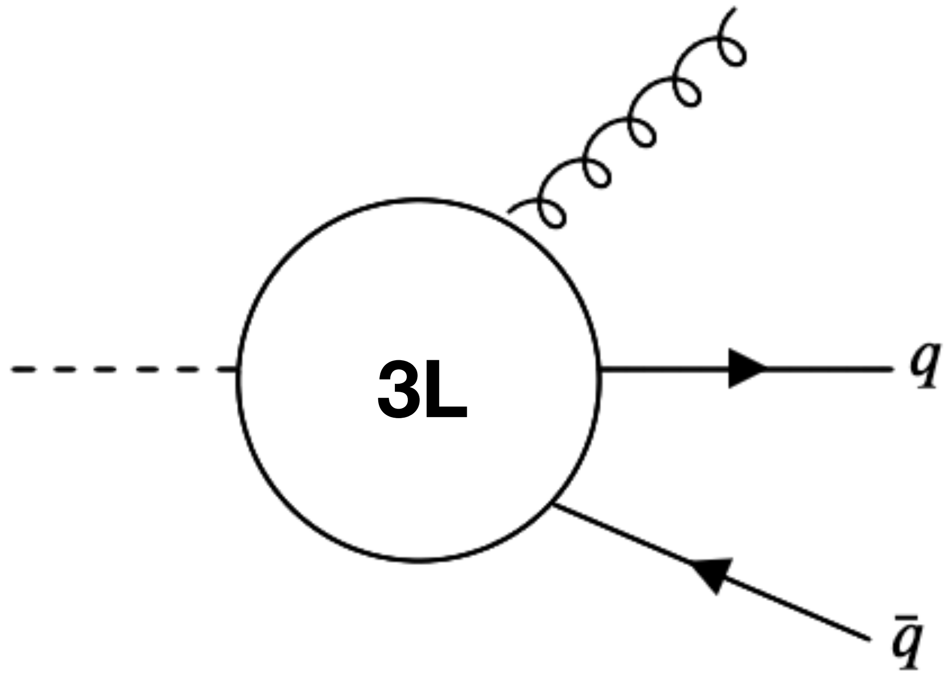
## (1) Phenomenology (Higgs plus jet)

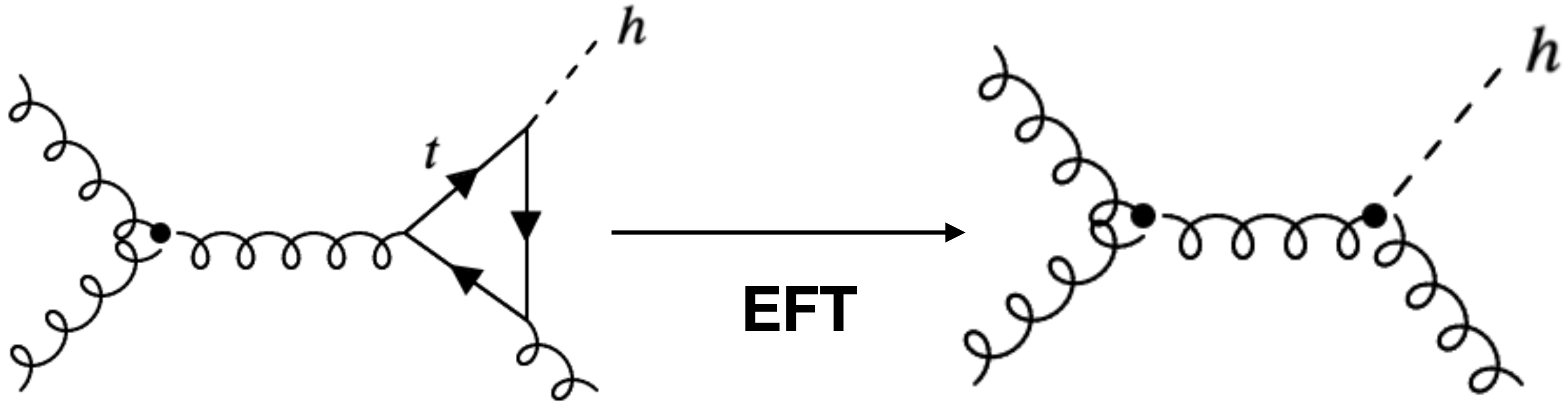



$H_{ggg}$ **amplitudes**




$H_{gq\bar{q}}$ **amplitudes**



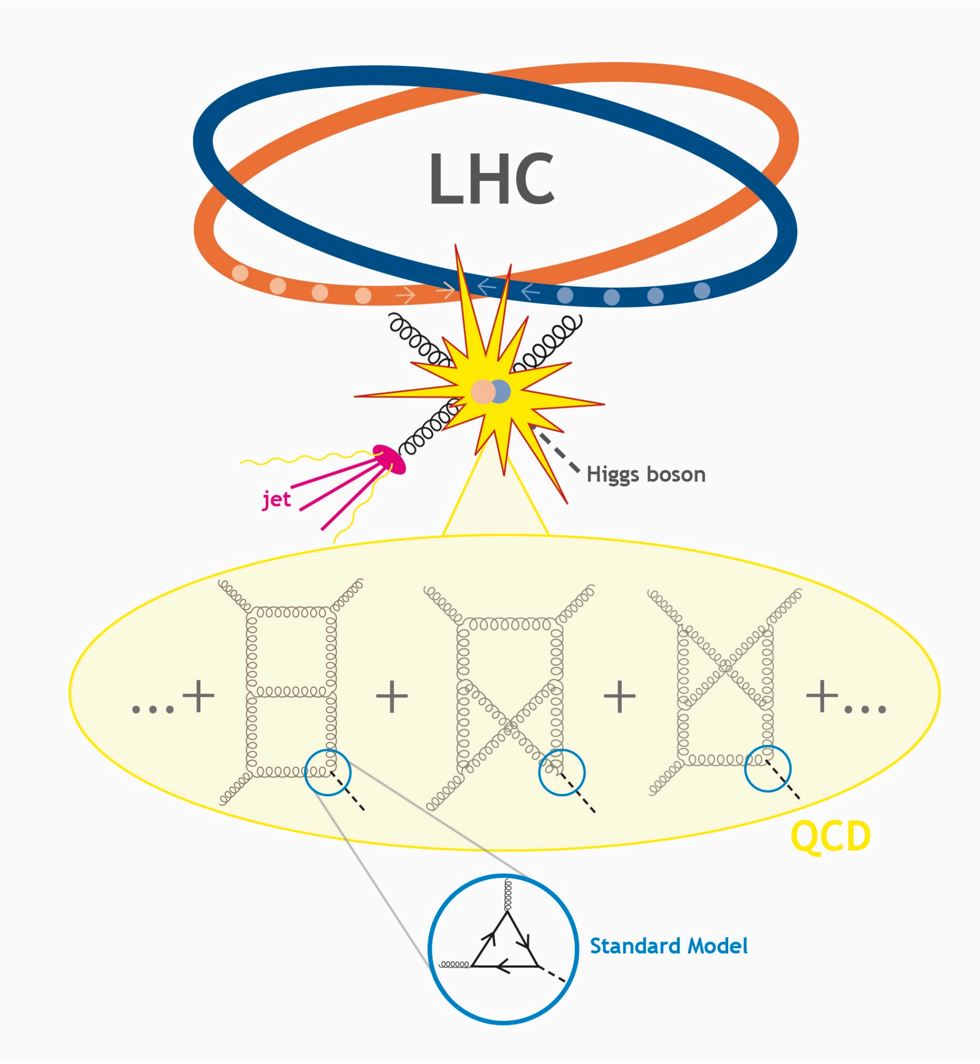

mediated by top quark

For percentage level precision, we need <u>N3LO</u>

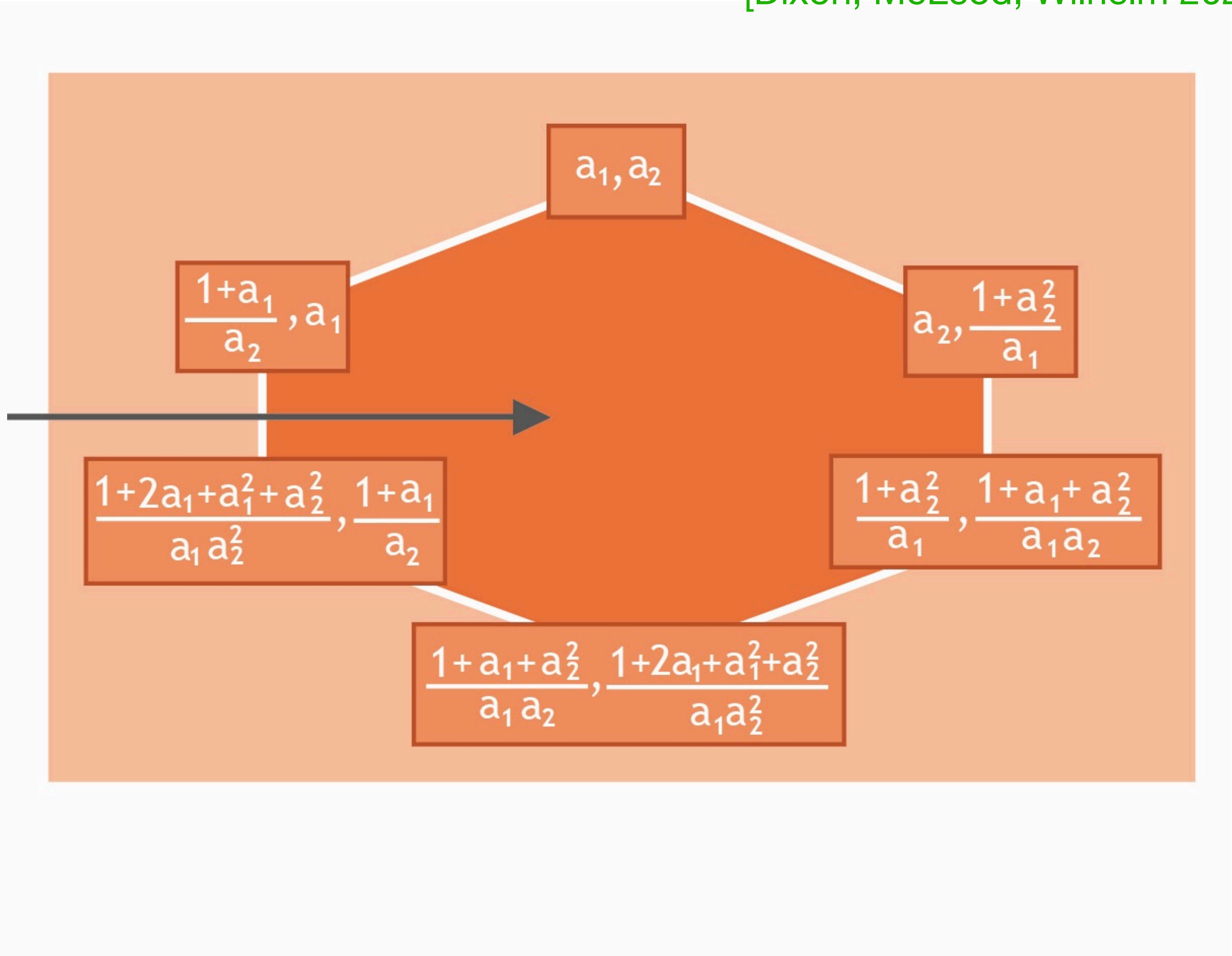
# Motivation

## (2) Hidden structure

- $C_2$ cluster algebra, extended-Steinmann-like condition [Chicherin, Henn, Papathanasiou 2020] [Dixon, McLeod, Wilhelm 2020]



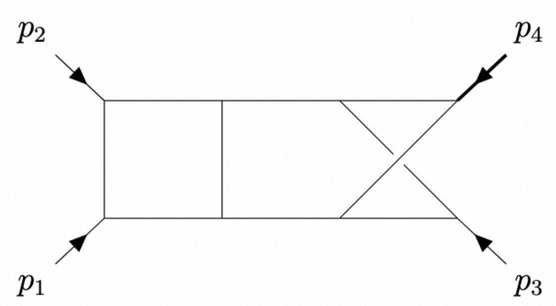

www.scattering-amplitudes.com



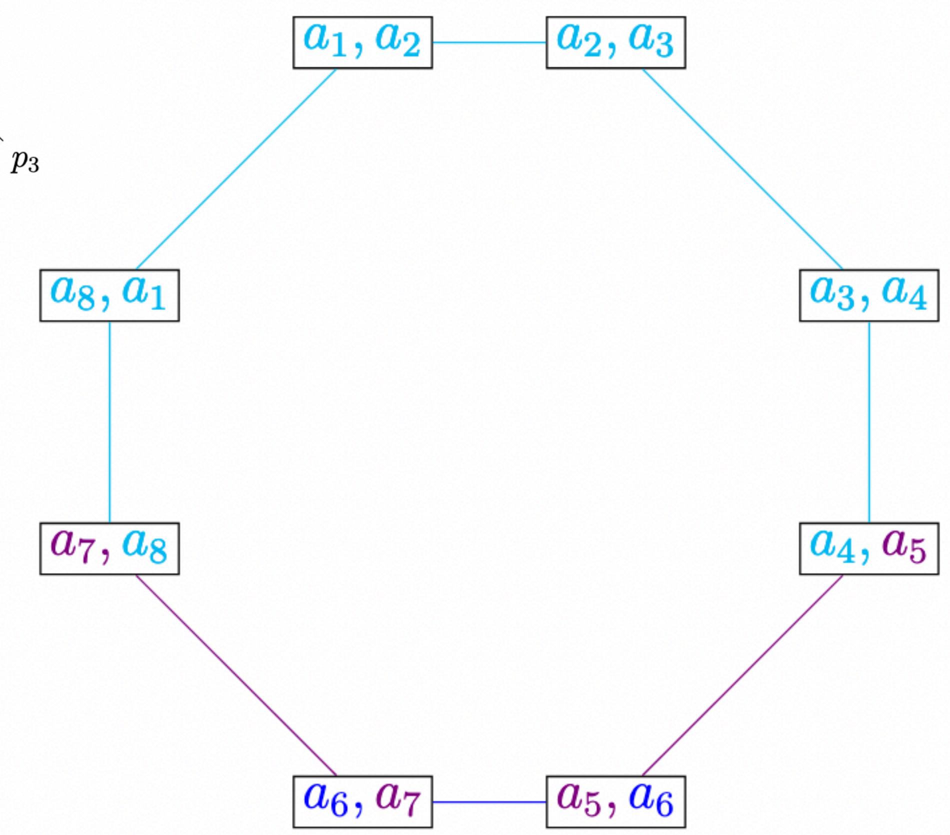

$$\alpha = \{z_1, z_2, 1 - z_1 - z_2, 1 - z_1, 1 - z_2, z_1 + z_2\}$$

$$\tilde{A}_i \cdot \tilde{A}_j = 0 \Longrightarrow \ldots \otimes \cancel{\alpha_i \otimes \alpha_j} \otimes \ldots$$ for $i, j \in \{4,5,6\}$ with $i \neq j$

$G_2$ cluster algebra




[Aliaj, Papathanasiou 2024]



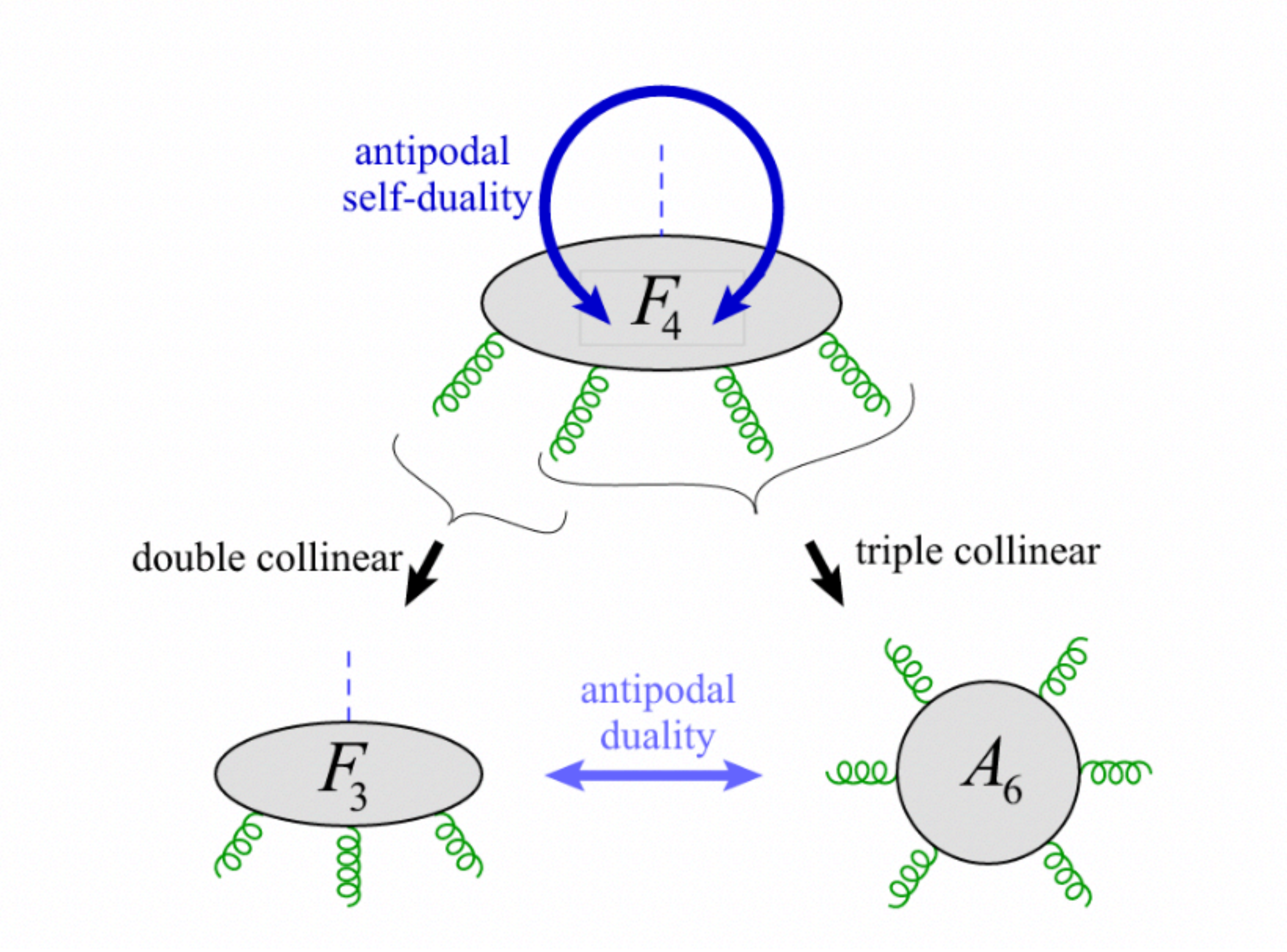

$$\alpha = \{z_1, z_2, 1 - z_1 - z_2, 1 - z_1, 1 - z_2, z_1 + z_2, 1 - 2z_1 + z_1^2 - z_2, z_1 - z_1^2 - z_2\}$$

# Motivation

## (2) Hidden structure

- Antipodal duality

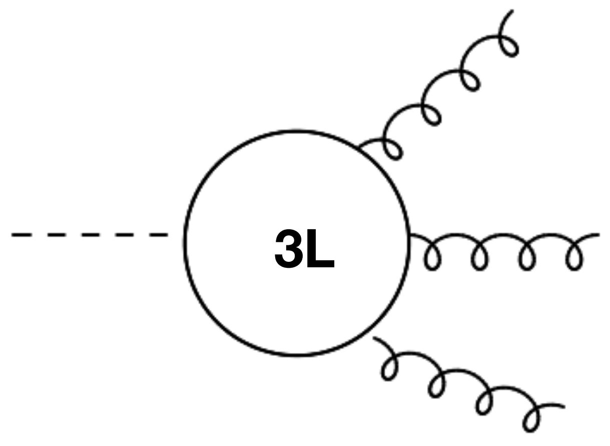
[Dixon, Gürdoğan, Liu, McLeod, Wihlem 2022]




[Dixon, Gürdoğan, McLeod, Wihlem 2021]

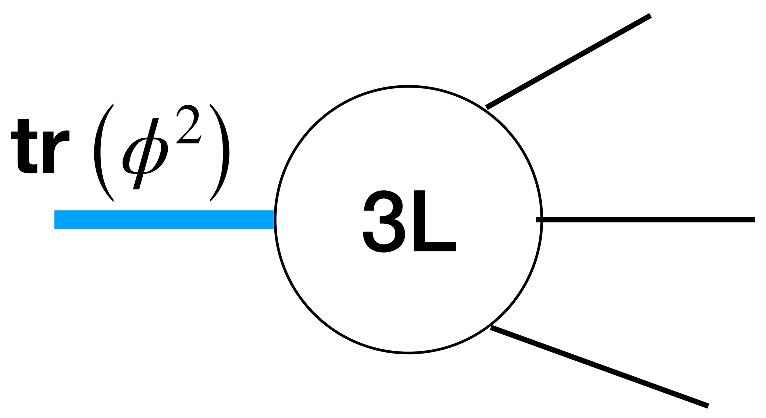
- Maximal transcendentality conjecture

$H_{ggg}$ **amplitudes**




**Maximally transcendental part**

**3-point tr $\left(\phi^2\right)$ form factor in N=4 SYM**




[Brandhuber, Travaglini, Yang 2012]

# Motivation

## (2) Hidden structure

**Textbook approach :** Integrals $\xrightarrow{\text{Insert}}$ Amplitudes $\xrightarrow{\text{Observe}}$ Physics

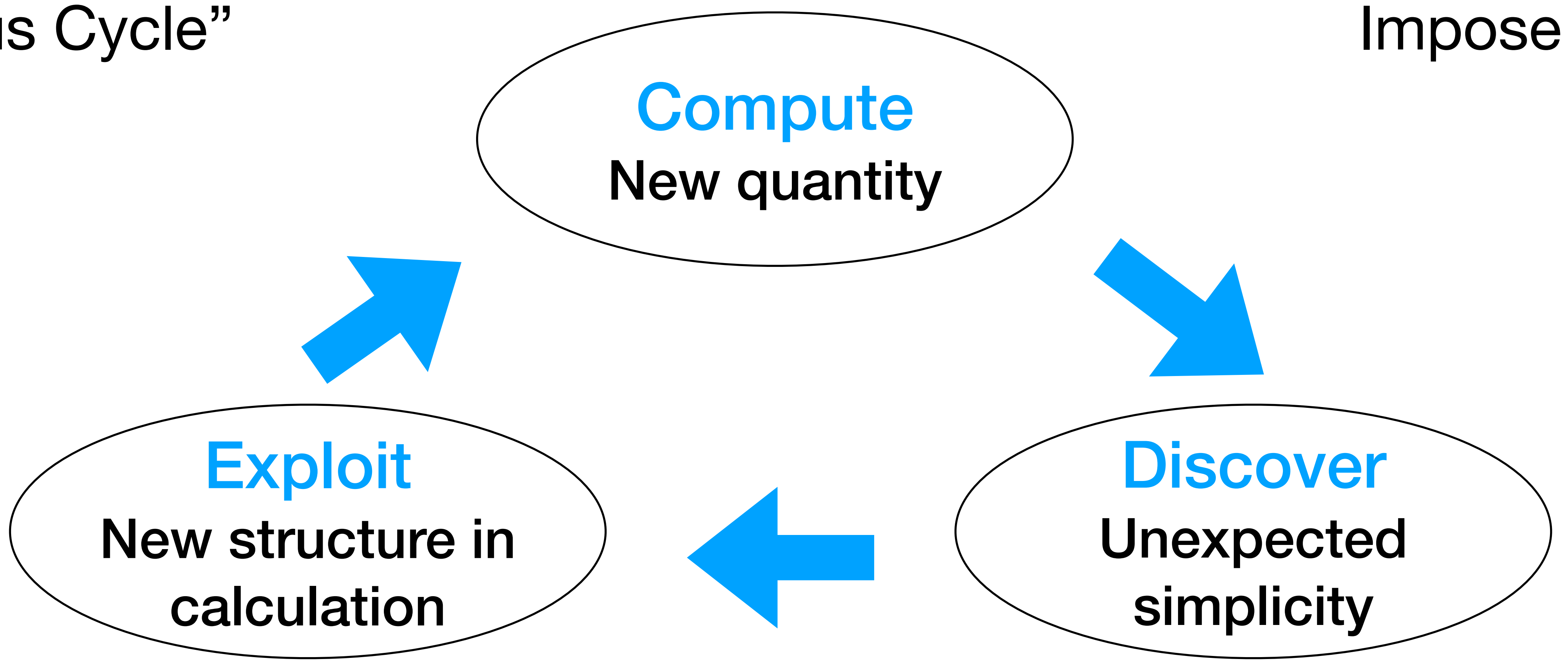
**Bootstrapping approach :** ~~Integrals~~ Amplitudes $\xleftarrow{\text{Enforce}}$ Physics

**Grading functions approach :**

$$\begin{tikzcd} \text{Integrals} \arrow[r, leftrightarrow, "\text{Tie}"] & \text{Amplitudes} & \text{Physics} \arrow[ll, bend left, "\text{Impose}"] \end{tikzcd}$$

[Jakubčík @HP2 2024]

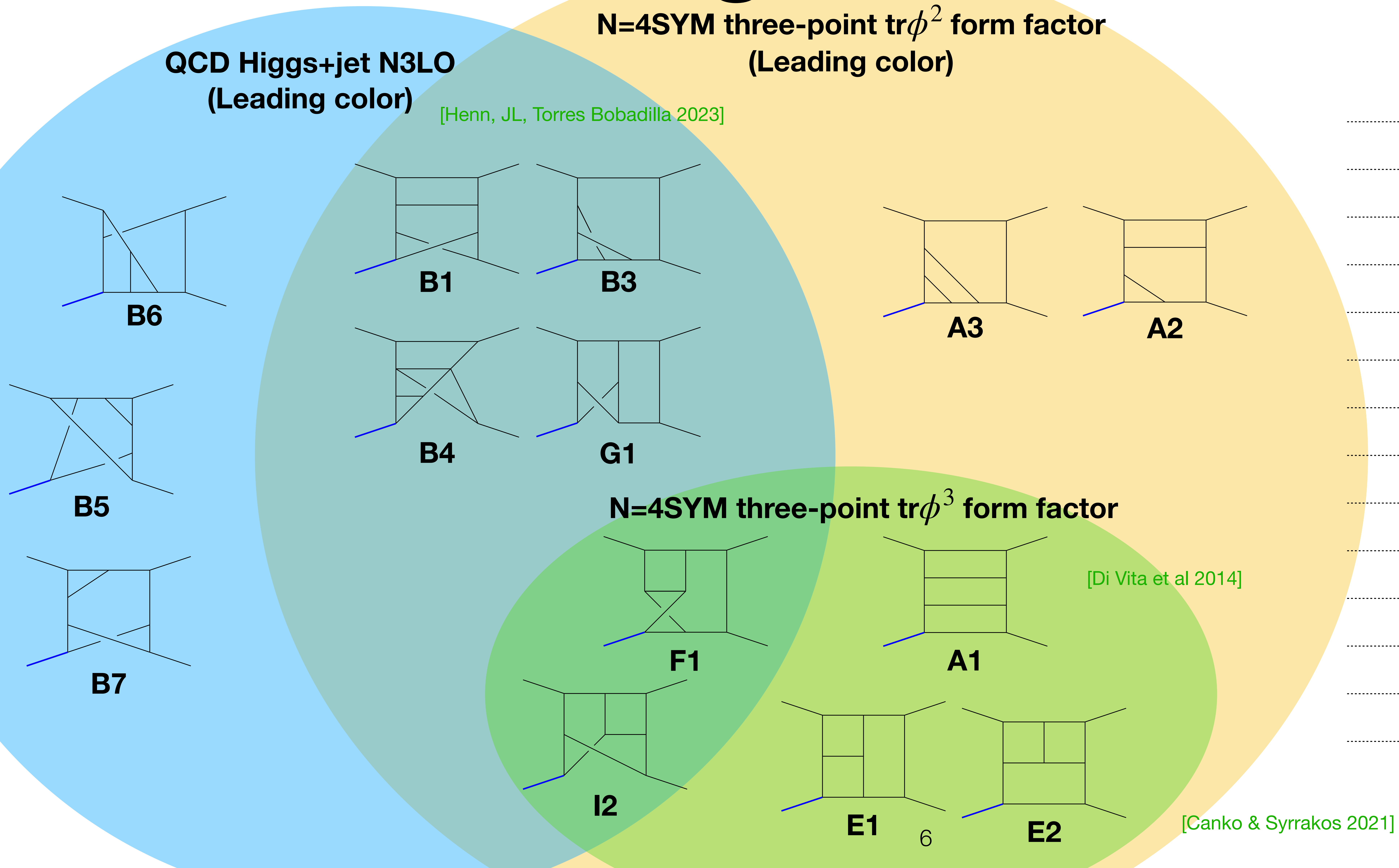
The "Virtuous Cycle"




[Bern's talk in Snowmass 2022]
[McLeod's talk in Galaxy meets QCD 2024]

[Gehrmann, Henn, Jakubčík, JL, Mella, Syrrakos, Tancredi, Torres Bobadilla 2024]

# Integral Families




| | # MI |
|---|---|
| A1 | 83 |
| A2 | 100 |
| A3 | 80 |
| B1 | 150 |
| B3 | 90 |
| B4 | 143 |
| B5 | 70 |
| B6 | 150 |
| B7 | 89 |
| E1 | 166 |
| E2 | 117 |
| F1 | 214 |
| G1 | 254 |
| I2 | 305 |

(Family : reducible top sector)

# Differential equation method

## Building canonical differential equations

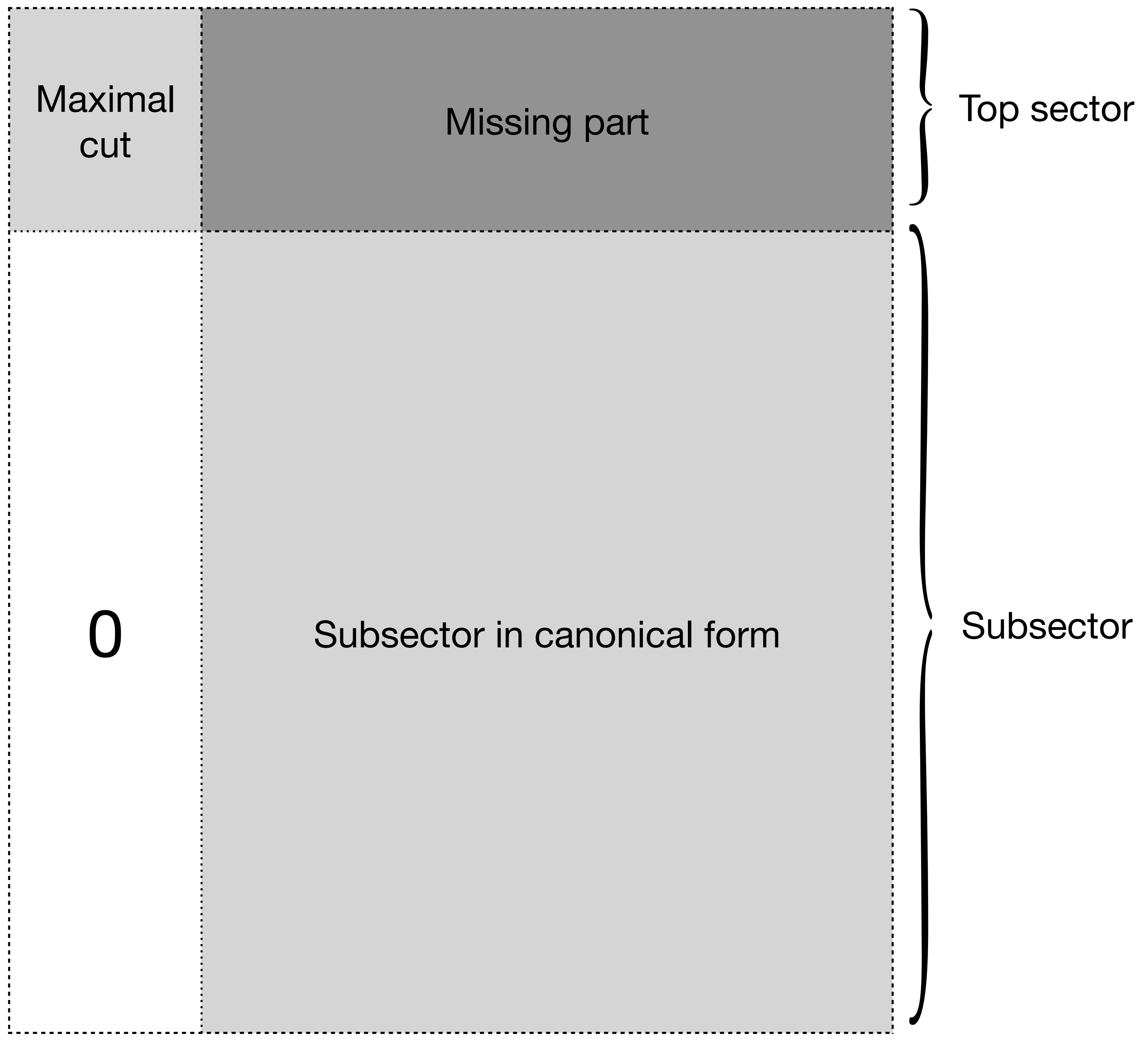
[Henn 2013]

Good choice of basis for Feynman integrals can significantly simplify the computation of differential equation.

$$d\vec{f}\left(\vec{x},\epsilon\right) = \epsilon\left(d\tilde{A}\right)\vec{f}\left(\vec{x};\epsilon\right), \text{ with } \tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right]$$

- Subsector : DlogBasis, Mapping from other families, loop-by-loop approach
  [Wasser 2022] [Flieger, Torres Bobadilla, 2022]

- Topsector : Matrix rotation, loop-by-loop approach




By FiniteFlow, Kira

[Peraro, 2019] [Klappert, Lange, Maierhöfer, Usovitsch, 2020]

# Iterated integrals

$\vec{f}\left(\overrightarrow{x}, \epsilon\right) = \mathbb{P}\exp\left[\epsilon \int_\gamma d\tilde{A}\right]\vec{f}_0(\epsilon)$ where $\vec{f}_0(\epsilon)$ is a boundary vector

$\gamma * \left(\omega_i\right) = k_i(t)\, dt$ function $k_i$ are defined by pulling back the 1-form $\omega_i$ to the interval [0,1]

An ordinary line integral is given by $\int_\gamma \omega_1 = \int_{[0,1]} \gamma * \left(\omega_1\right) = \int_0^1 k_1\left(t_1\right) dt_1$

Iterated integral of $\omega_1 \ldots \omega_n$ along $\gamma$ is defined by $\int_\gamma \omega_1 \ldots \omega_n = \int_{0 \le t_1 \le \ldots \le t_n \le 1} k_1\left(t_1\right) dt_1 \ldots k_n\left(t_n\right) dt_n$

[Chen 1977]

If the alphabet is rational functions , one can write the answer in terms of Goncharov polylogarithms

$$G(\overrightarrow{a}_n; z) \equiv G(\overrightarrow{a}_1, \overrightarrow{a}_{n-1}; z) \equiv \int_0^z \frac{dt}{t - a_1} G(\overrightarrow{a}_{n-1}; t)$$

with $G\left(a_1; z\right) = \int_0^z \frac{dt}{t - a_1}$ and $G(\overrightarrow{0}_n; z) \equiv \frac{1}{n!}\log^n(z)$

# Function space of 3loop integrals

$$\vec{\alpha} = \{p_4^2, s, t, p_4^2 - s - t, p_4^2 - s, p_4^2 - t, s + t, \frac{(p_4^2 - s - t)s - R}{(p_4^2 - s - t)s + R}, \frac{st - R}{st + R}, p_4^4 - t\left(p_4^2 + s\right), p_4^4 - s\left(p_4^2 + t\right),\ t^2 + p_4^2(s - t), s^2 - p_4^2(s - t), -p_4^2 t + \left(p_4^2 - s\right)^2\}$$

with $R = \sqrt{-p_4^2 s(p_4^2 - s - t)t}$

**20 letters (including kinematic crossings)**

**Simple planar letters**

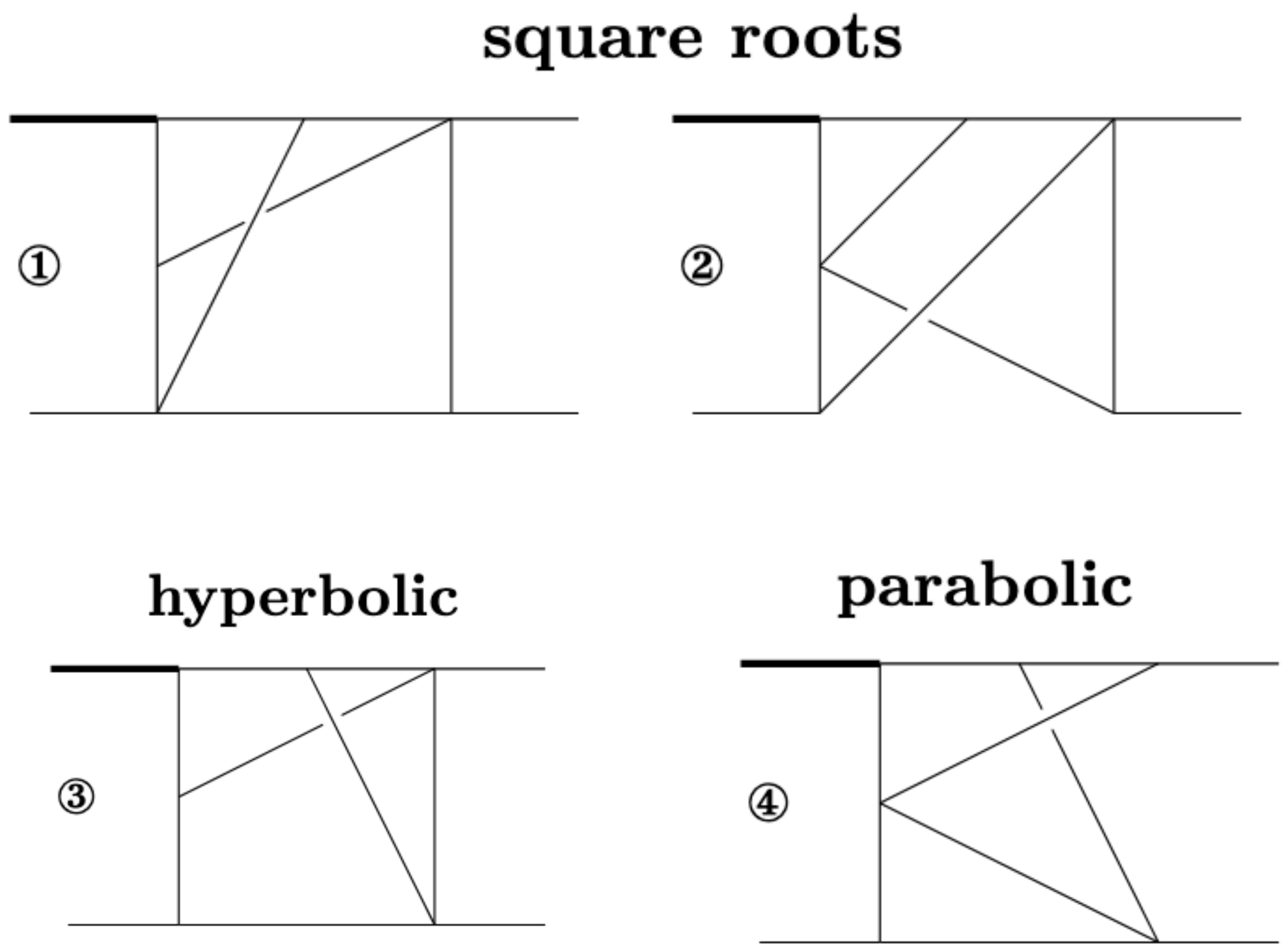
$$\{x, y, z, 1 - x, 1 - y, 1 - z\}$$

$\longrightarrow\ l_{1-6}$

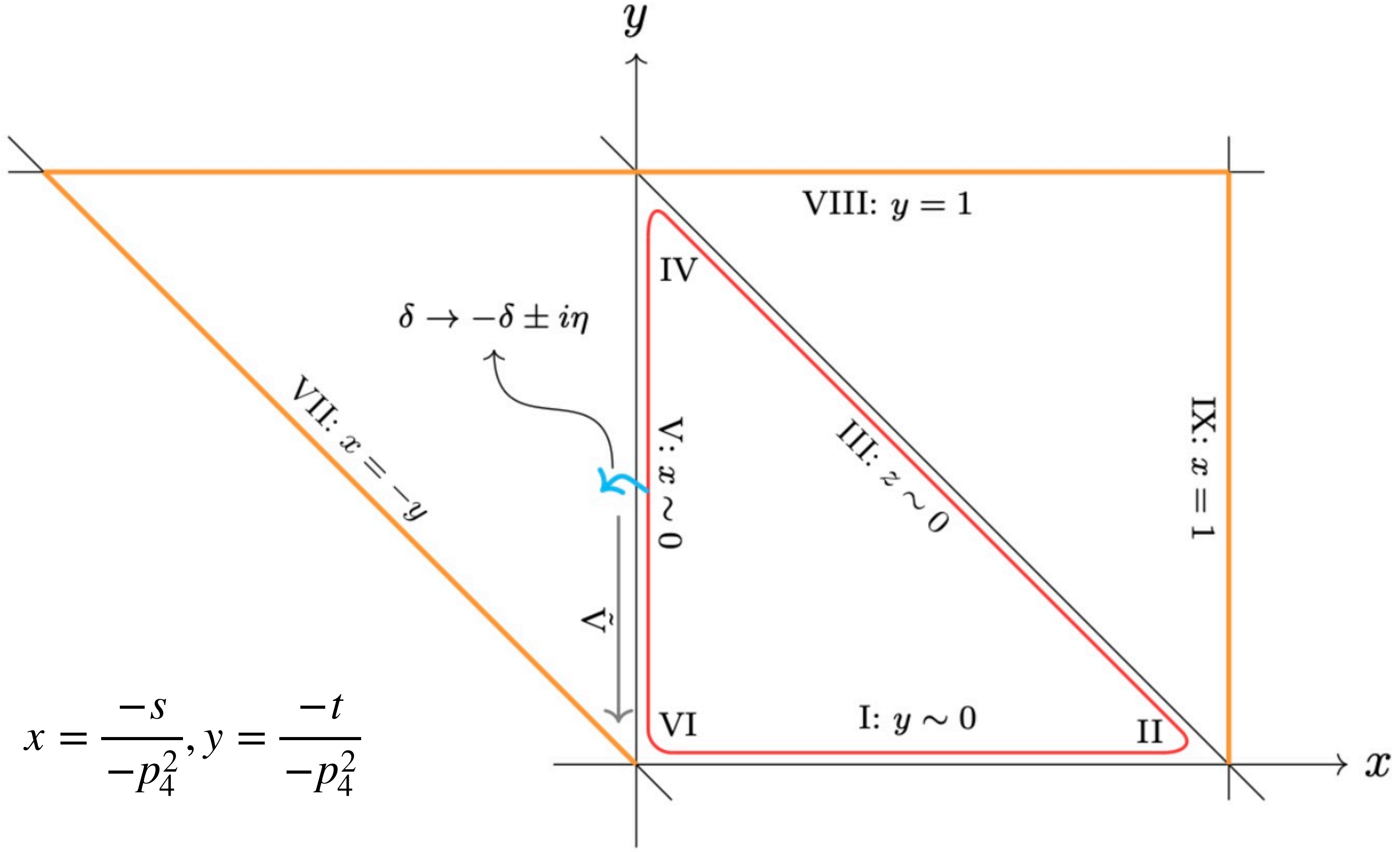
$$x = \frac{-s}{-p_4^2}, y = \frac{-t}{-p_4^2}, z = \frac{-p_4^2 + s + t}{-p_4^2}$$

**New letter types**

- Sqaure root letter : $\dfrac{xy - \sqrt{xyz}}{xy + \sqrt{xyz}}$ $\quad l_{7-8}$
- Hyperbolic letter : $x^2 + xy + y$ $\quad l_{9-10}, l_{17-20}$
- Parabolic letter : $x^2 - x + y$ $\quad l_{11-16}$



# Fixing boundary constants




$$x = \frac{-s}{-p_4^2}, y = \frac{-t}{-p_4^2}$$

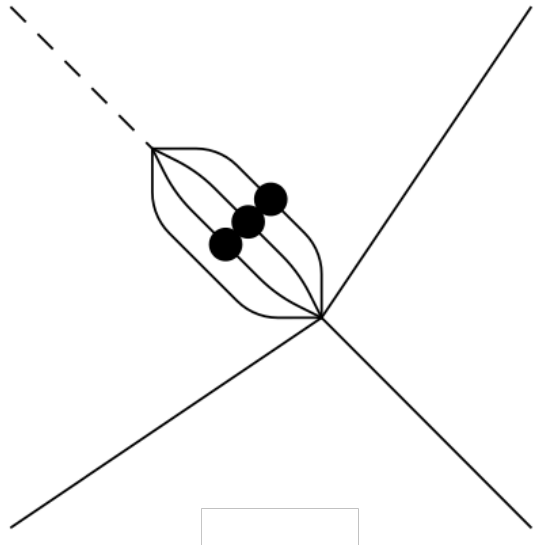
| segment | start-/end-point | $x(\delta,t)$ | $y(\delta,t)$ |
|---|---|---|---|
| I | $P_1 \to P_2$ | $t$ | $\delta$ |
| II | $P_2 \to P_3$ | $1-\delta\left[(1-t)^2+t^2\right]$ | $\delta t^2$ |
| III | $P_3 \to P_4$ | $1-t-\delta$ | $t$ |
| IV | $P_4 \to P_5$ | $\delta(1-t)^2$ | $1-\delta\left[(1-t)^2+t^2)\right]$ |
| V | $P_5 \to P_6$ | $\delta$ | $1-t$ |
| VI | $P_6 \to P_7$ | $\delta t^2$ | $\delta(1-t)^2$ |
| VII | $P_7 \to P_8$ | $-t(1-\delta)$ | $t$ |
| VIII | $P_8 \to P_9$ | $(-1+t)\cup t$ | $1-\delta$ |
| IX | $P_9 \to P_{10}$ | $1-\delta$ | $1-t$ |

At each segment, your "effective" alphabet is only rational. $\vec{\alpha}_t = \{t, t-\frac{1}{2}, t-1, t-\frac{e^{i\pi/4}}{\sqrt{t}}, t-\frac{e^{-i\pi/4}}{\sqrt{t}}\}$

At each segment, one can impose the constraint with the information of singularities.

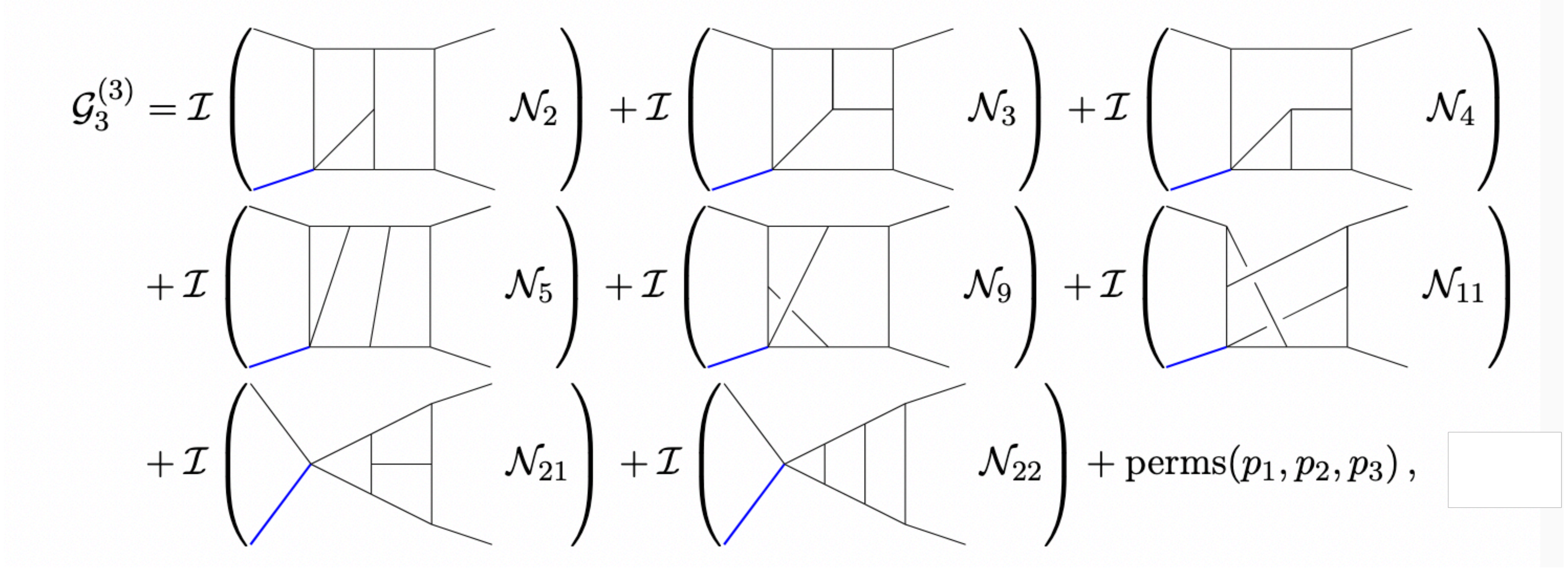
By matching, one can relate the boundary vector in one segment to another.

➡ Fix all the boundary constants up to one integral.



# Computing tr$\phi^3$ form factor

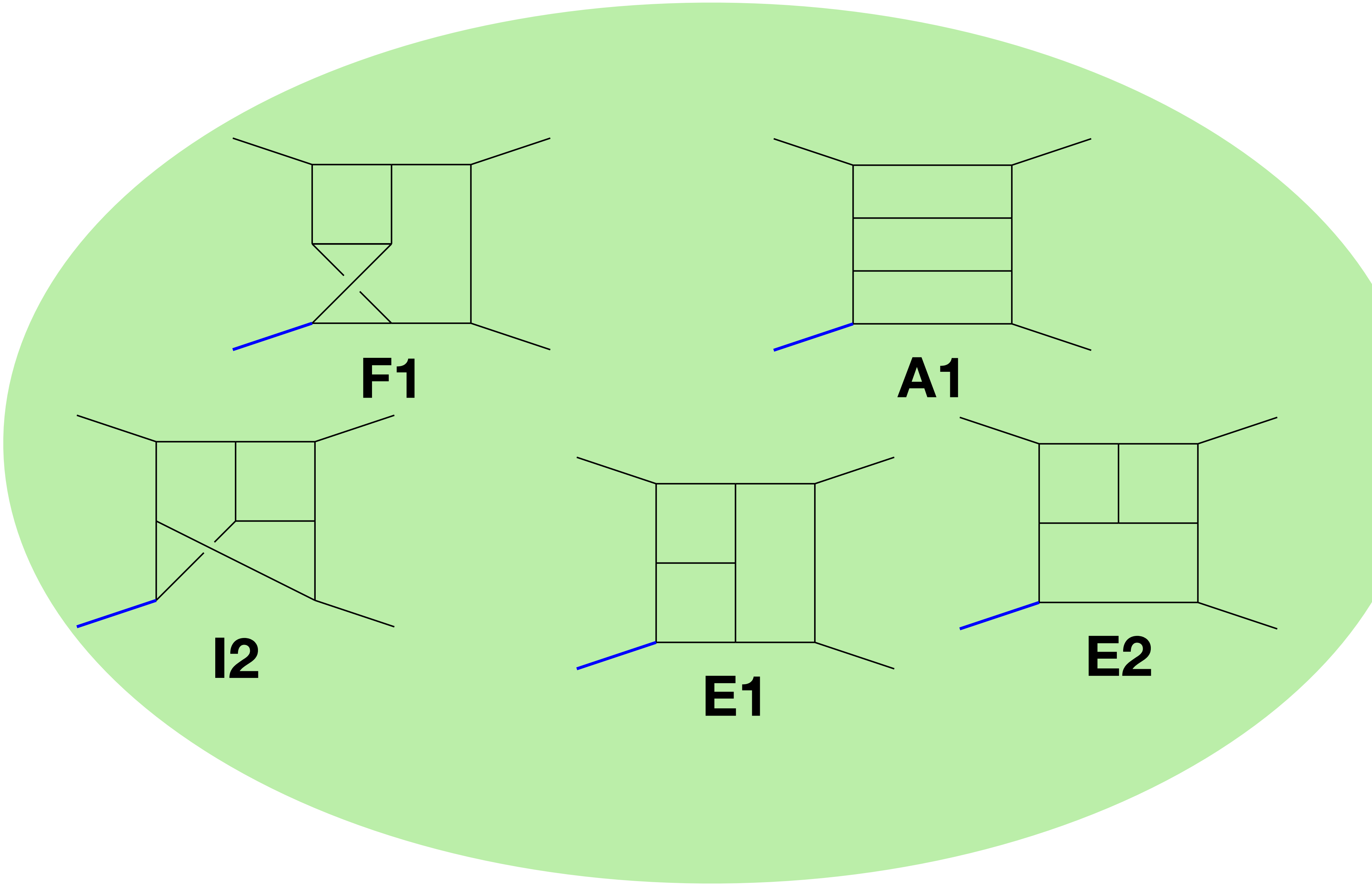
[Henn, JL, Torres Bobadilla 2024]




[Lin, Yang, Zhang 2021]

**IBP reduction**

[Peraro, 2019]



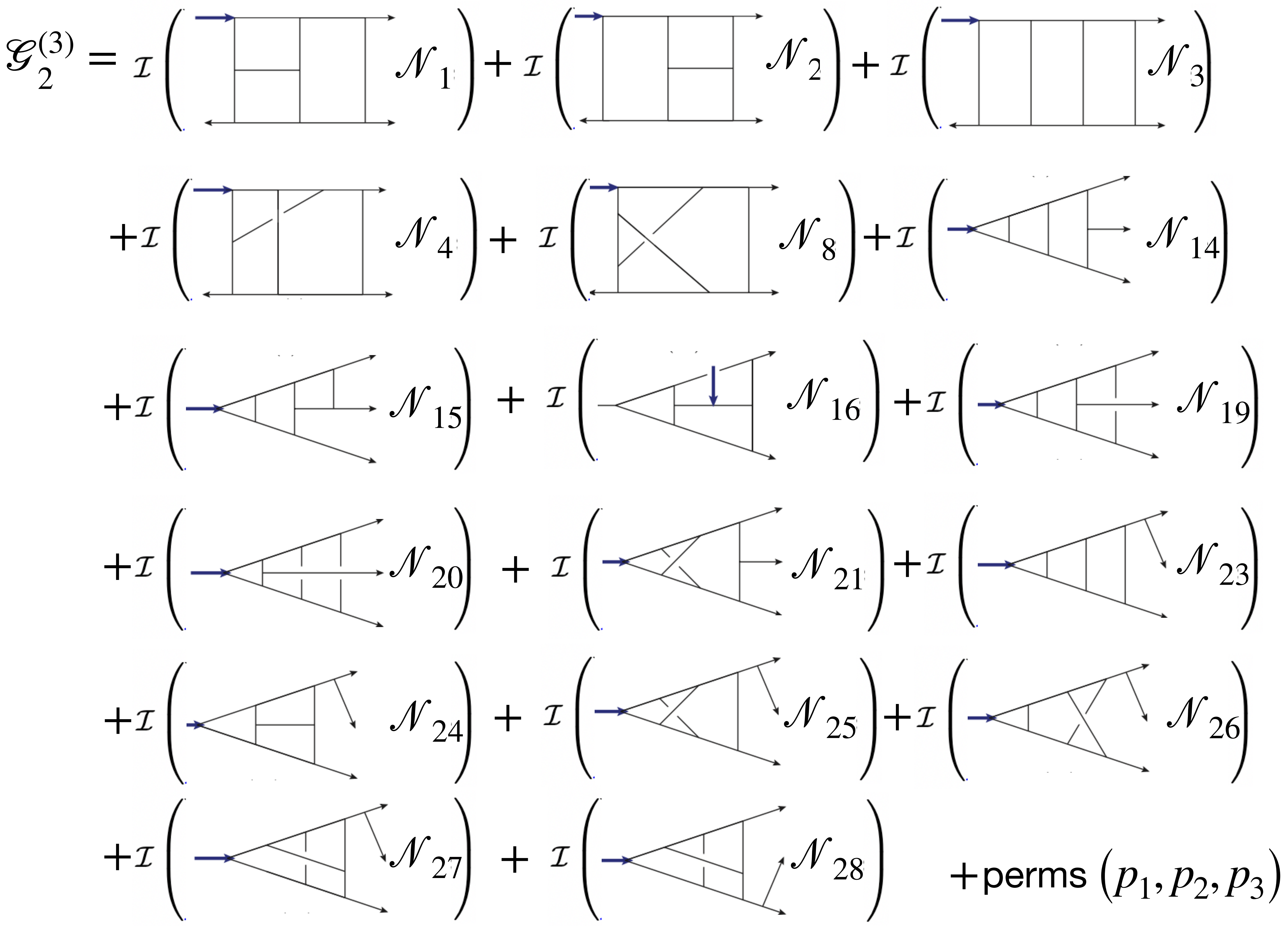

$$\mathcal{G}_3^{(3)} = c_1 ut[Ax123{,}1] + c_2 ut[Ax123{,}2] + \ldots$$

$c_1, c_2, \ldots$ are rational numbers!

| | Ordering | | | | | | |
|---|---|---|---|---|---|---|---|
| Family | 123 | 132 | 213 | 231 | 312 | 321 | Total |
| A | 78 | 73 | 46 | 71 | 44 | 40 | 352 |
| E1 | 12 | 10 | 11 | 10 | 11 | 10 | 64 |
| E2 | 23 | 20 | 18 | 19 | 18 | 14 | 112 |
| F1 | 32 | 26 | 28 | 22 | 23 | 19 | 150 |
| I2 | 2 | 0 | 0 | 0 | 0 | 0 | 2 |

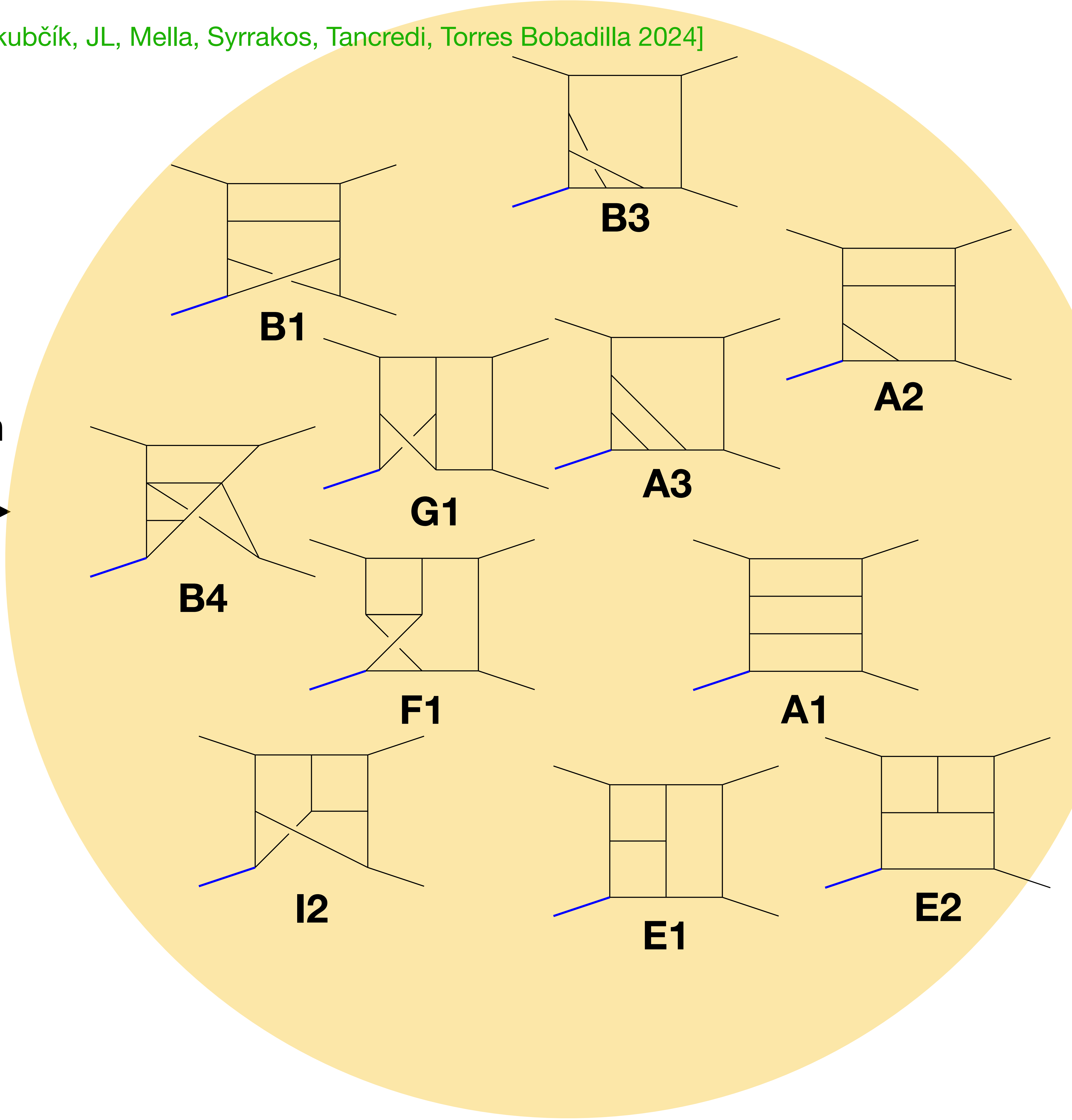
# Computing tr$\phi^2$ form factor

[Gehrmann, Henn, Jakubčík, JL, Mella, Syrrakos, Tancredi, Torres Bobadilla 2024]




**IBP reduction**

[Peraro, 2019]




[Lin, Yang, Zhang 2021]

$$\mathcal{G}_2^{(3)} = c_1 ut[A\text{x}123,1] + c_2 ut[A\text{x}123,2] + \ldots$$

$c_1, c_2, \ldots$ are rational numbers!

# Minimal space and grading the functions

**Our function space is the vector space of transcendental function**

We solve differential equations in canonical form and the solution can be expressed by Chen iterated integrals

$$I\left(\omega_1, \ldots, \omega_n; \vec{x}\right) = \int_\gamma \omega_1 \omega_2 \cdots \omega_n, \qquad I\left(; \vec{x}\right) = 1,$$

where $\omega_i = \omega_i\left(\vec{x}\right)$ are differential forms in the kinematic invariants and
$\gamma = \gamma\left(\vec{1}_0, \vec{x}\right)$ is a curve connecting the base point $\vec{1}_0$ to a generic kinematic point $\vec{x}$.

$\longrightarrow$ **Length n iterated integral has transcendental weight n**

Same definition can be extended to transcendental number $\xi_n = \pi^2, \zeta_n, \ldots,$ which correspond to special values of the iterated integrals, and we also assign weight −1 to ϵ.

Ex) $st^2 J_{E1;1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}$

$$= \frac{2}{9} + \epsilon\left(-\frac{2}{3}I\left(\omega_1\right) - \frac{2}{3}I\left(\omega_2\right)\right) + \epsilon^2\left(\frac{8\pi^2}{27} - \frac{2}{3}I\left(\omega_1, \omega_4\right) + 2I\left(\omega_1, \omega_1\right) + 2I\left(\omega_1, \omega_2\right) + 2I\left(\omega_2, \omega_1\right) - \frac{2}{3}I\left(\omega_2, \omega_5\right) + 2I\left(\omega_2, \omega_2\right)\right) + \mathcal{O}\left(\epsilon^3\right)$$

# Minimal space and grading the functions

## Master integrals are not the minimal basis

The basis of our space is $b_{\mathcal{T}_\omega} = \{\epsilon^{-a}\xi_n^b I\left(\omega_1, \ldots, \omega_c; \overrightarrow{x}\right)\}$, with weight $w = a + nb + c$

Scattering amplitudes truncated at an ϵ order containing functions of weight at most $w$ are combinations of some set of transcendental functions with algebraic functions as prefactors,

$$\boxed{A_\omega = \sum a\left(\overrightarrow{x}\right) \cdot b_{\mathcal{T}_\omega}.}$$

$$S = \begin{array}{c|c} \begin{array}{c} J_1 \\ J_2 \\ \vdots \\ J_{n_{MI}} \end{array} & \begin{pmatrix} \; & & & \\ & & Q & \\ & & & \end{pmatrix} \\ & \begin{array}{cccc} 1 & \epsilon I(l_1) \cdots & \epsilon^2 I(l_1, l_2) & \cdots \end{array} \end{array}$$

Not all the master integrals are independent $\longrightarrow$ $\text{Rank}\,(S) \leq n_{\text{MI}} \leq \dim\left(\mathcal{T}_\omega\right)$

We can write the full transformation $T'$ as an invertible matrix over numbers,

$$\overrightarrow{J}_\omega = T' \cdot \begin{pmatrix} b_{\mathcal{M}_\omega} \\ \overrightarrow{0}_\omega \end{pmatrix}$$

$b_{\mathcal{M}_\omega}$ : Rank $(S)$ vector

[Jakubčík @HP2 2024]

# Minimal space and grading the functions

**Further minimization using physical properties**

$$\vec{J}_{\omega} = T' \cdot \begin{pmatrix} b_{\mathscr{M}_\omega} \\ \vec{0}_{\omega} \end{pmatrix}$$

$b_{\mathscr{M}_\omega}$ : Rank $(S)$ vector

$$\mathscr{M}_{\omega} = A_{\parallel,\omega} \oplus A_{\perp,\omega}$$

Physical Unphysical

**Grading : Organizing the functions in $\mathscr{M}_\omega$ so that $A_{\parallel,\omega}$ and $A_{\perp,\omega}$ does not mixed**

All the components of $b_{\mathscr{M}_\omega}$ are independent. $\longrightarrow$ No further cancellation between the functions.

We can remove the component that violates the constraints from physics.

$$\begin{pmatrix} \vec{\psi}_w \\ \vec{0}_w \end{pmatrix} = \begin{pmatrix} T'' & \\ & \mathbb{I} \end{pmatrix} \cdot (T')^{-1} \cdot \vec{J}_w$$
$$\equiv T \cdot \vec{J}_w \,,$$

# Minimal space and grading the functions

**Higgs plus jet amplitudes in leading color**

13812 seemingly different canonical combination

$\downarrow$

1282 unique combination of functions

- The leading color terms in the color expansion are

$$N^3,\ N_f N^2,\ N_f^2 N \text{ and } N_f^3$$

- The terms proportional to $N_f^2$ and $N_f^3$ contain only planar letter $l_{1-6}$

$\longrightarrow$ The function space can be simplified to $\psi_{94-1282}$.

| | $\psi_1$ $-\psi_3$ | $\boldsymbol{\psi_4}$ $\boldsymbol{-\psi_9}$ | $\psi_{10}$ $-\psi_{18}$ | $\psi_{19}$ $-\psi_{24}$ | $\boldsymbol{\psi_{25}}$ $\boldsymbol{-\psi_{36}}$ | $\psi_{37}$ $-\psi_{51}$ | $\psi_{52}$ $-\psi_{63}$ | $\boldsymbol{\psi_{64}}$ $\boldsymbol{-\psi_{93}}$ | $\psi_{94}$ $-\psi_{111}$ | $\psi_{112}$ $-\psi_{1281}$ | $\boldsymbol{\psi_{1282}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| equals FF in sYM | | | | | | | | | | | ✓ |
| violates (3.13) | | | * | * | | * | * | | ✓ | | |
| satisfies (3.14, 3.16) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| $l_{7-20}$ appears | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | | | |
| • from $\mathcal{O}(\epsilon^4)$ | ✓ | ✓ | | | | | | | | | |
| $\hookrightarrow$ only parabolic | | ✓ | | | | | | | | | |
| $\hookrightarrow$ also roots | ✓ | | | | | | | | | | |
| • from $\mathcal{O}(\epsilon^5)$ | | | ✓ | ✓ | ✓ | | | | | | |
| $\hookrightarrow$ only parabolic | | | | | ✓ | | | | | | |
| $\hookrightarrow$ also hyperbolic | | | ✓ | ✓ | | | | | | | |
| $\hookrightarrow$ also roots | | | ✓ | | | | | | | | |
| • from $\mathcal{O}(\epsilon^6)$ | | | | | | ✓ | ✓ | ✓ | | | |
| $\hookrightarrow$ only parabolic | | | | | | | | ✓ | | | |
| $\hookrightarrow$ also hyperbolic | | | | | | ✓ | ✓ | | | | |
| $\hookrightarrow$ also roots | | | | | | ✓ | | | | | |

# Minimal space and grading the functions

### Higgs plus jet amplitudes in leading color

- The $N^3$ , $N_f N^2$ terms contain non-planar topologies.

  $\longrightarrow$ We tried the numerical reduction to see the function space to all functions $\psi_{1-1282}$ (with 8 propagators cut)

**Ex) Hggg amplitudes**

$$
\begin{aligned}
\alpha^{(3)}|_{N^3} = \epsilon^{-6} \Big[ &+ \frac{1746}{48841}\epsilon^2\,\psi_4 + \frac{22}{289}\epsilon^2\,\psi_6 + \frac{10}{169}\epsilon^2\,\psi_7 \\
&+ \frac{11}{18}\Big(\frac{37}{3}\epsilon\,\psi_{25} + 5\epsilon\,\psi_{26} + 6\epsilon\,\psi_{27} + \frac{11}{2}\epsilon\,\psi_{28} + 8\epsilon\,\psi_{29} \\
&- 7\epsilon\,\psi_{30} - 5\epsilon\,\psi_{31} - \frac{5}{2}\epsilon\,\psi_{32} + 2\epsilon\,\psi_{33} - \frac{22}{3}\epsilon\,\psi_{34} + 2\epsilon\,\psi_{35}\Big)\Big] \\
&+ \text{terms with letters } l_{1-6} \text{ only} \\
&+ \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

| | $\psi_1$ $-\psi_3$ | $\boldsymbol{\psi_4}$ $\boldsymbol{-\psi_9}$ | $\psi_{10}$ $-\psi_{18}$ | $\psi_{19}$ $-\psi_{24}$ | $\boldsymbol{\psi_{25}}$ $\boldsymbol{-\psi_{36}}$ | $\psi_{37}$ $-\psi_{51}$ | $\psi_{52}$ $-\psi_{63}$ | $\boldsymbol{\psi_{64}}$ $\boldsymbol{-\psi_{93}}$ | $\psi_{94}$ $-\psi_{111}$ | $\psi_{112}$ $-\psi_{1281}$ | $\boldsymbol{\psi_{1282}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| equals FF in sYM | | | | | | | | | | | ✓ |
| violates (3.13) | | | * | * | | * | * | | ✓ | | |
| satisfies (3.14, 3.16) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| $l_{7-20}$ appears | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | | | |
| • from $\mathcal{O}(\epsilon^4)$ | ✓ | ✓ | | | | | | | | | |
| $\hookrightarrow$ only parabolic | | ✓ | | | | | | | | | |
| $\hookrightarrow$ also roots | ✓ | | | | | | | | | | |
| • from $\mathcal{O}(\epsilon^5)$ | | | ✓ | ✓ | ✓ | | | | | | |
| $\hookrightarrow$ only parabolic | | | | | ✓ | | | | | | |
| $\hookrightarrow$ also hyperbolic | | | ✓ | ✓ | | | | | | | |
| $\hookrightarrow$ also roots | | | ✓ | | | | | | | | |
| • from $\mathcal{O}(\epsilon^6)$ | | | | | | ✓ | ✓ | ✓ | | | |
| $\hookrightarrow$ only parabolic | | | | | | | | ✓ | | | |
| $\hookrightarrow$ also hyperbolic | | | | | | ✓ | ✓ | | | | |
| $\hookrightarrow$ also roots | | | | | | ✓ | | | | | |

# Result

## Higgs plus jet amplitudes in leading color

- New letters $\omega_{\text{new}}$ shouldn't appear in order less than $\epsilon^{2L}$ $\longrightarrow$ **Checked and later can be used for reconstruction**
- We also checked the square root letter and hyperbolic letter drops out in the finite remainder
- Only six parabolic letters survive among the new letters
- In $Hggg$ amplitude, no new letter appears at weight 6 (only appears at weight 4 & 5)

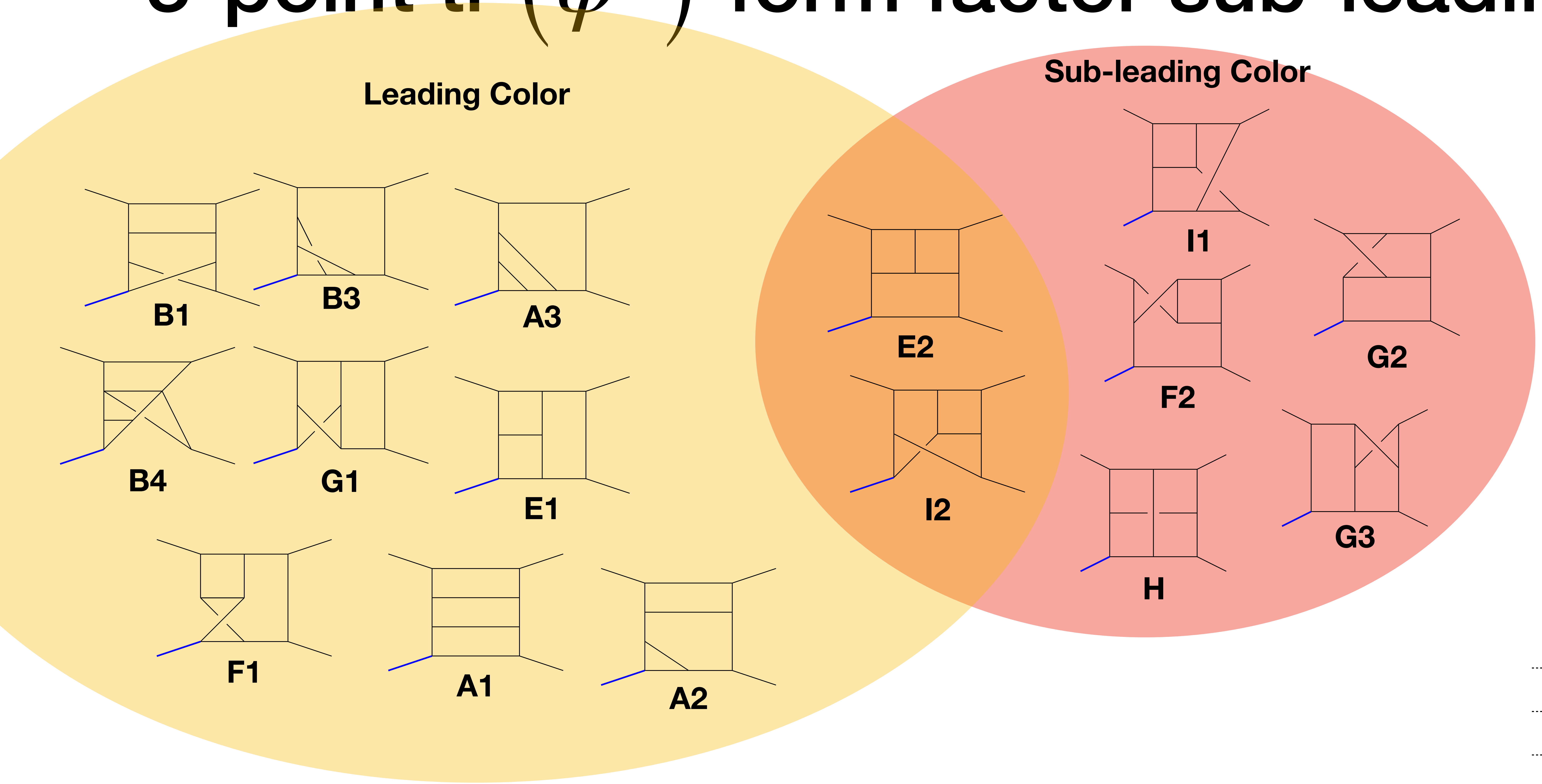
  $\longrightarrow$ **Hint to maximal transcendentality conjecture**

$$\alpha^{(3)}\Big|_{N^3} = \mathscr{G}_2^{(3)} + \sum_{i=1}^{1281} c_i \psi_i \text{ with } c_i = \mathcal{O}(\epsilon)$$

- In $Hgq\bar{q}$ amplitudes, the new parabolic letter appears also at weight 6
- Adjacency conditions hold at least for the functions containing the new letters $M_i \cdot M_j = M_j \cdot M_i = 0\,, \qquad i,j \in \{4,5,6\}$

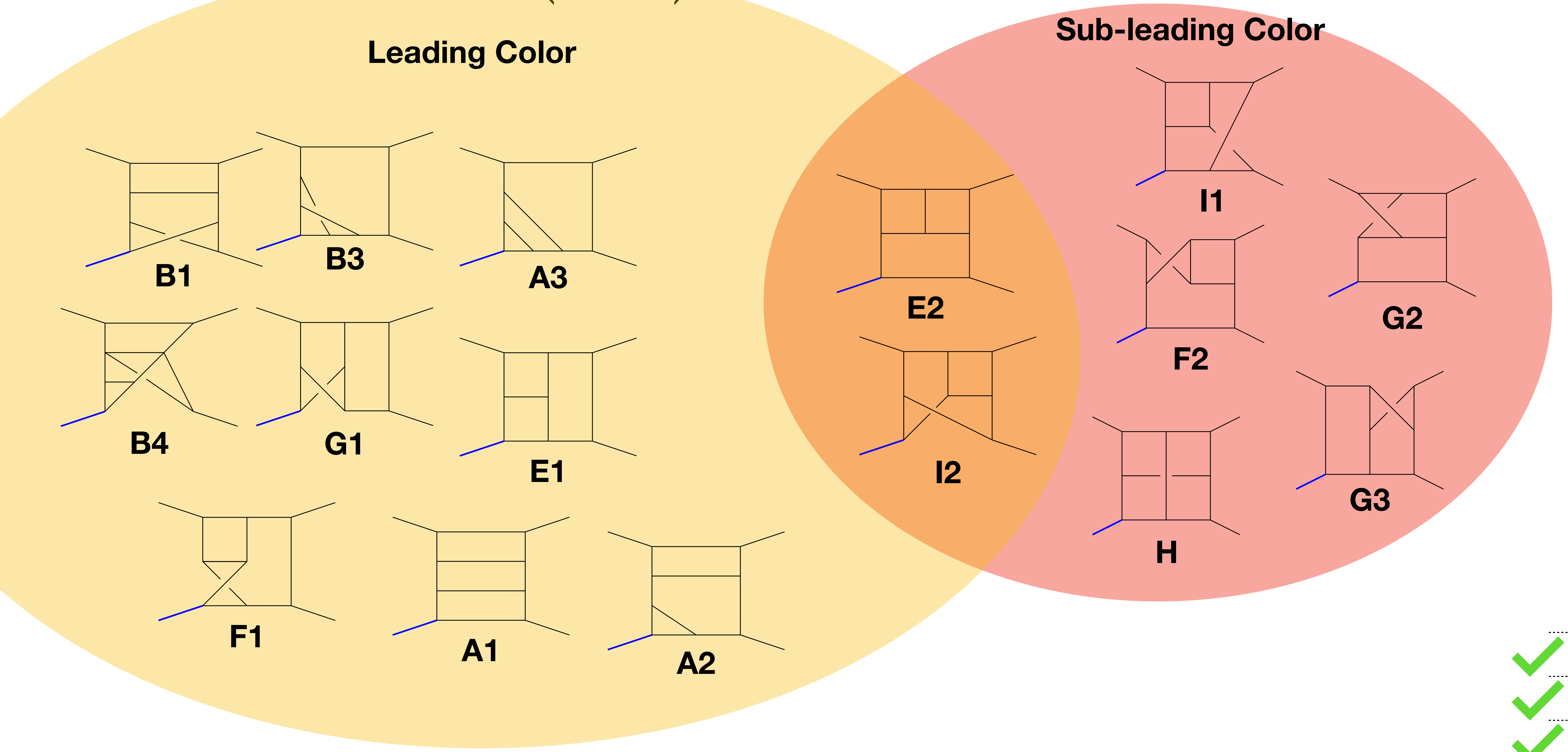
  $\longrightarrow$ **Potential for bootstrap approach?**

# 3-point tr $\left(\phi^2\right)$ form factor sub-leading color



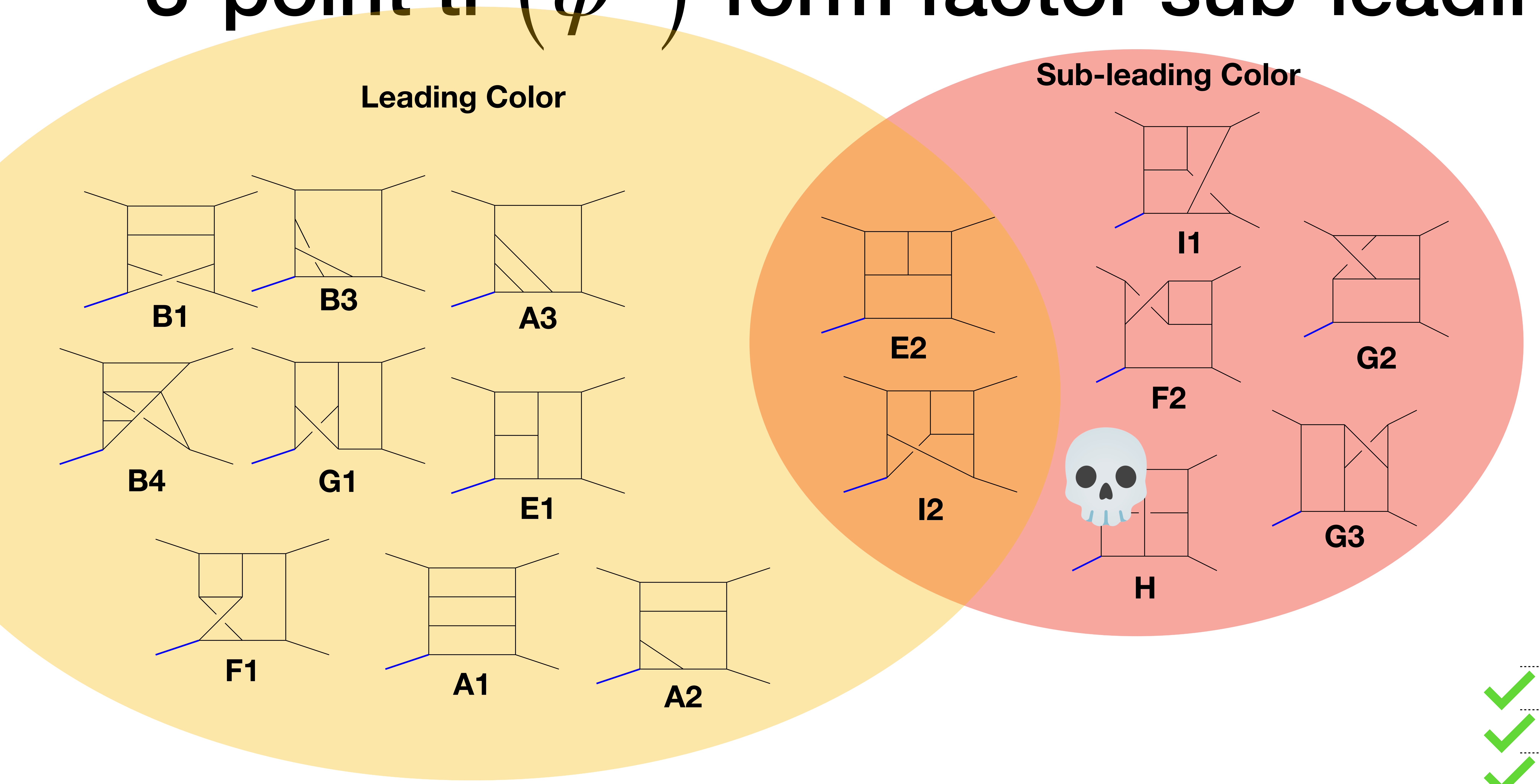

| | # MI(#topsector) |
|---|---|
| F2 | 136(4) |
| G2 | 174(4) |
| G3 | 176(4) |
| I1 | 277(8) |
| H | 371(19) |

# 3-point tr $\left(\phi^2\right)$ form factor sub-leading color



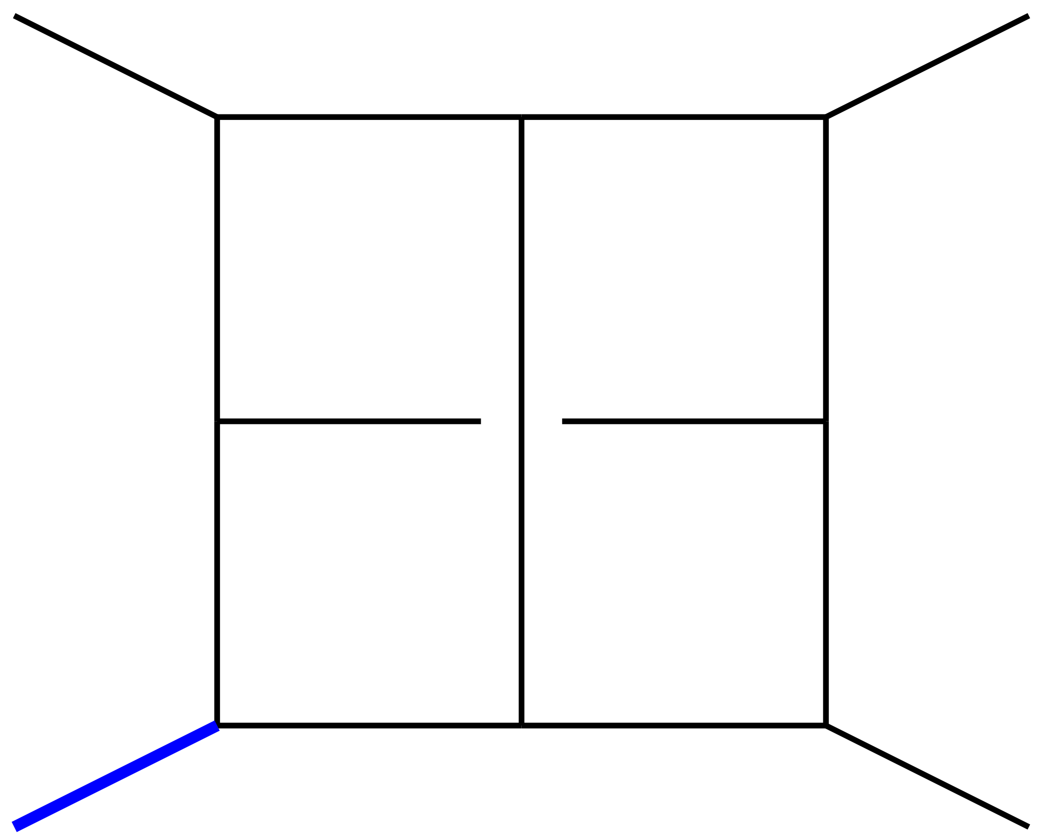

| | # MI(#topsector) |
|---|---|
| ✔ F2 | 136(4) |
| ✔ G2 | 174(4) |
| ✔ G3 | 176(4) |
| ✔ I1 | 277(8) |
| H | 371(19) |

# 3-point tr $\left(\phi^2\right)$ form factor sub-leading color

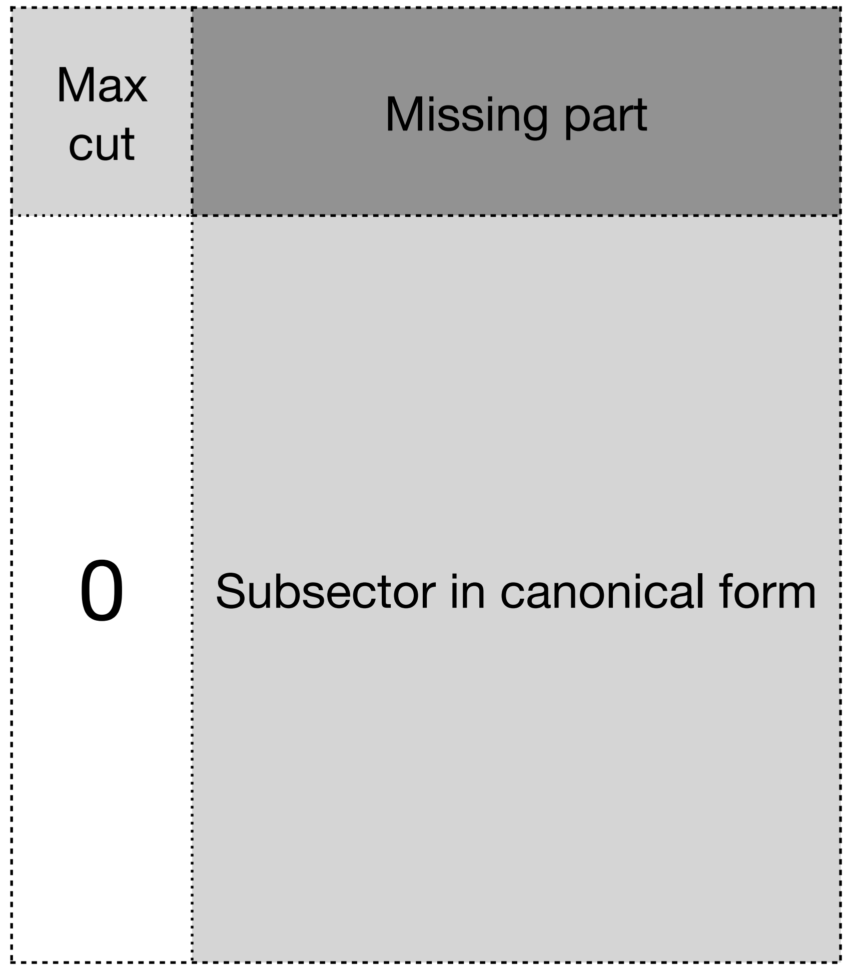



| | # MI(#topsector) |
|---|---|
| ✓ F2 | 136(4) |
| ✓ G2 | 174(4) |
| ✓ G3 | 176(4) |
| ✓ I1 | 277(8) |
| H | 371(19) |

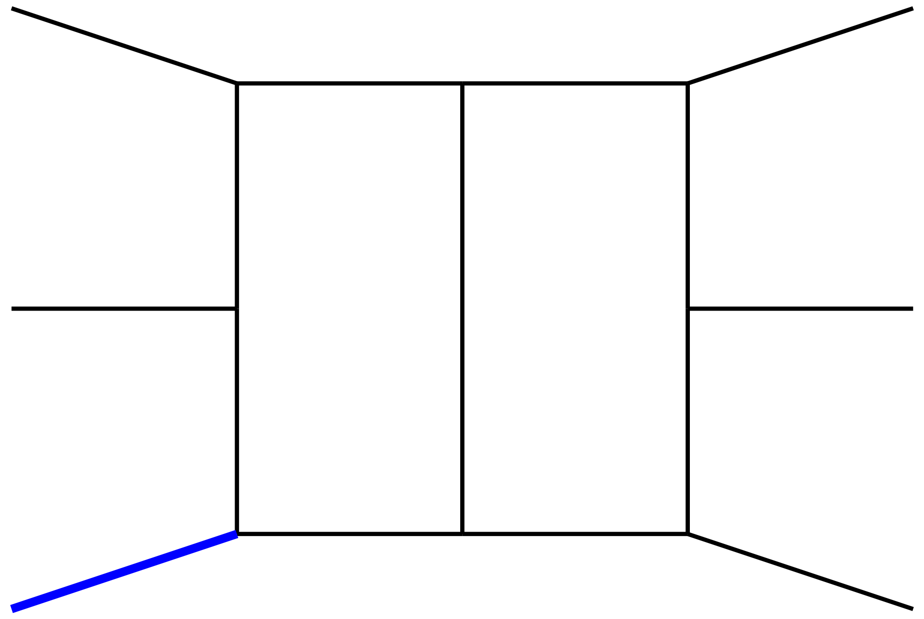
# Finding canonical basis of H

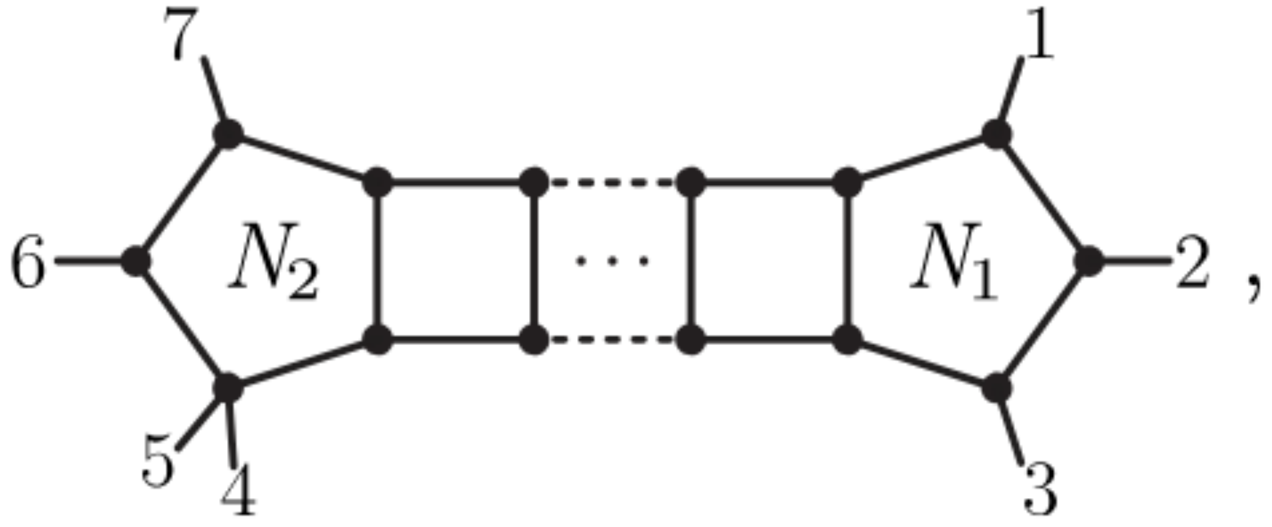


**Rotation of matrix**



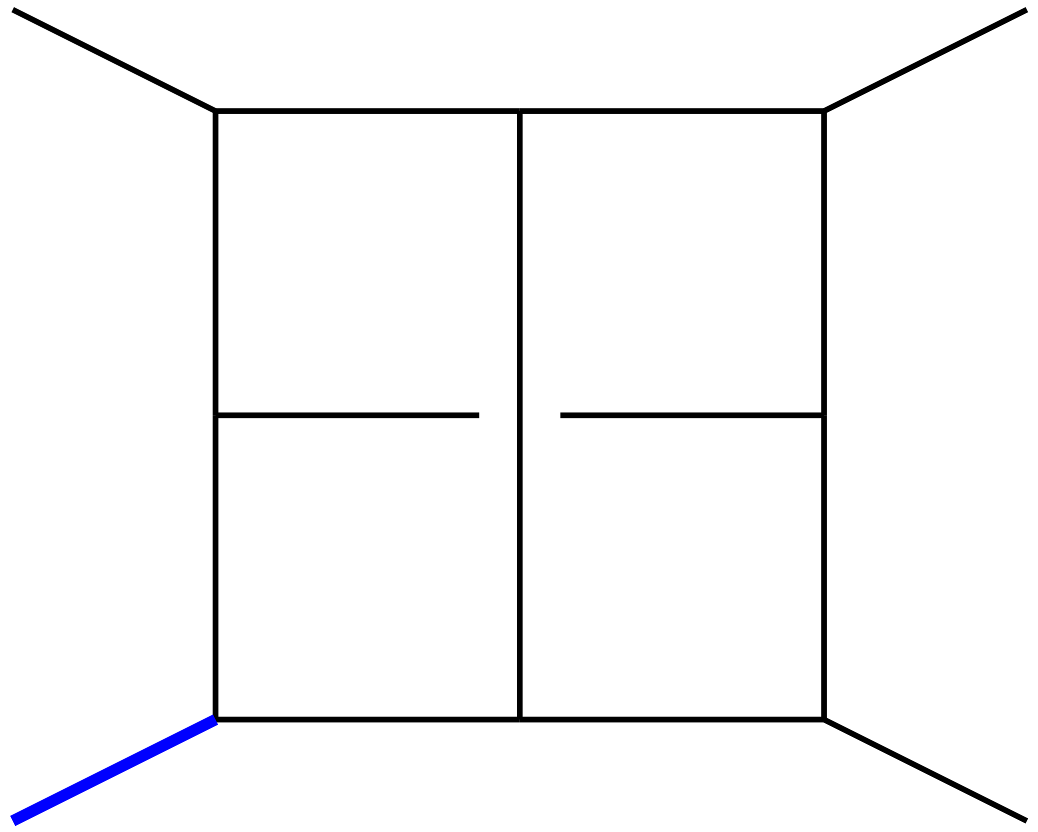

**N=4 sYM**



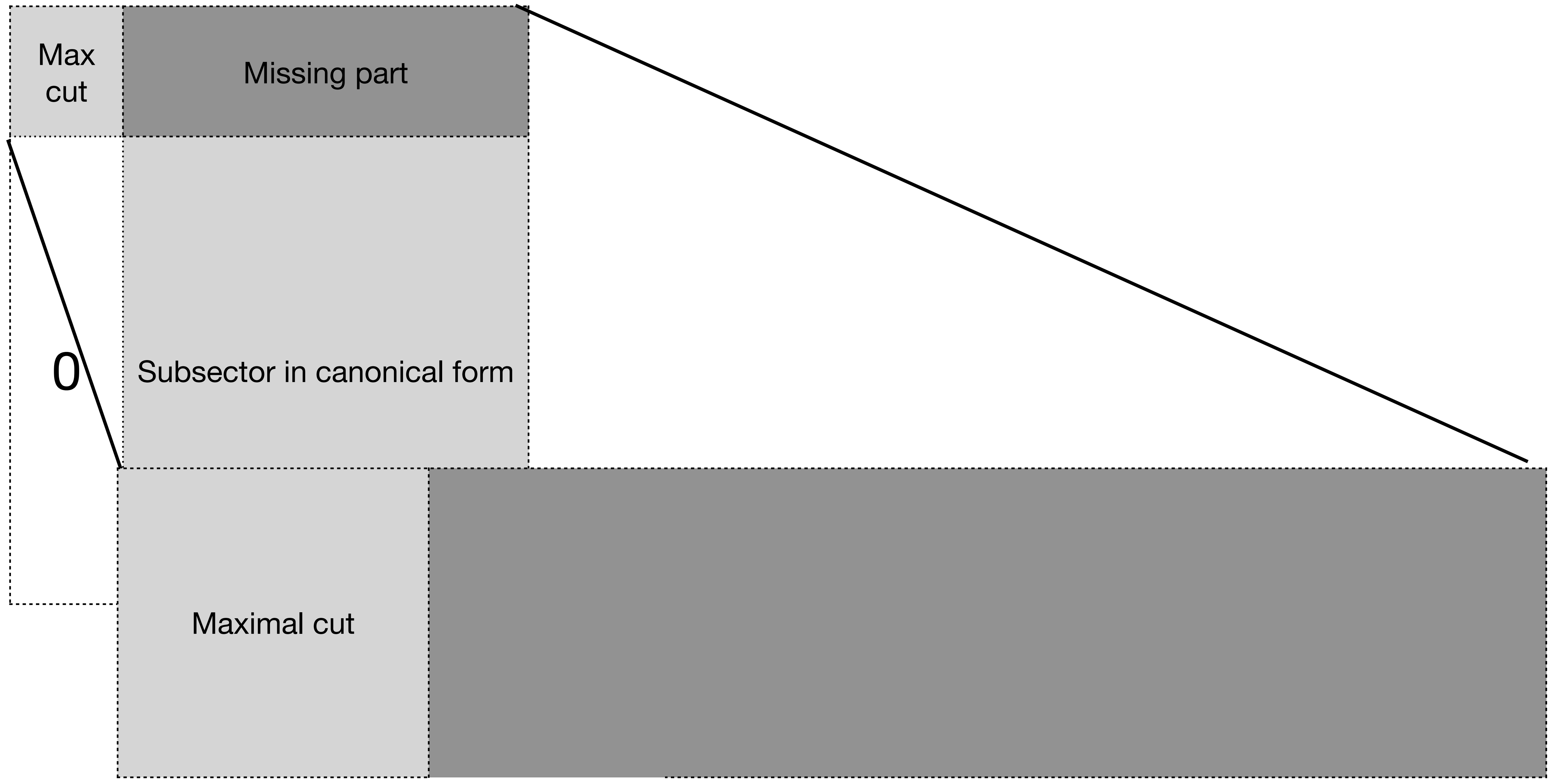



[Bourjaily, McLeod, von Hippel, Wilhelm 2018]

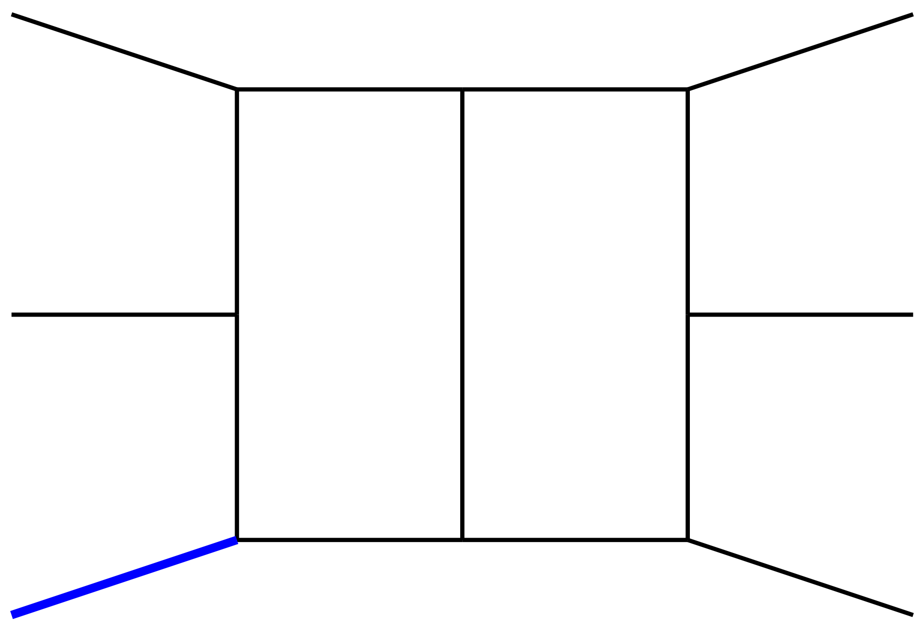
# Finding canonical basis of H

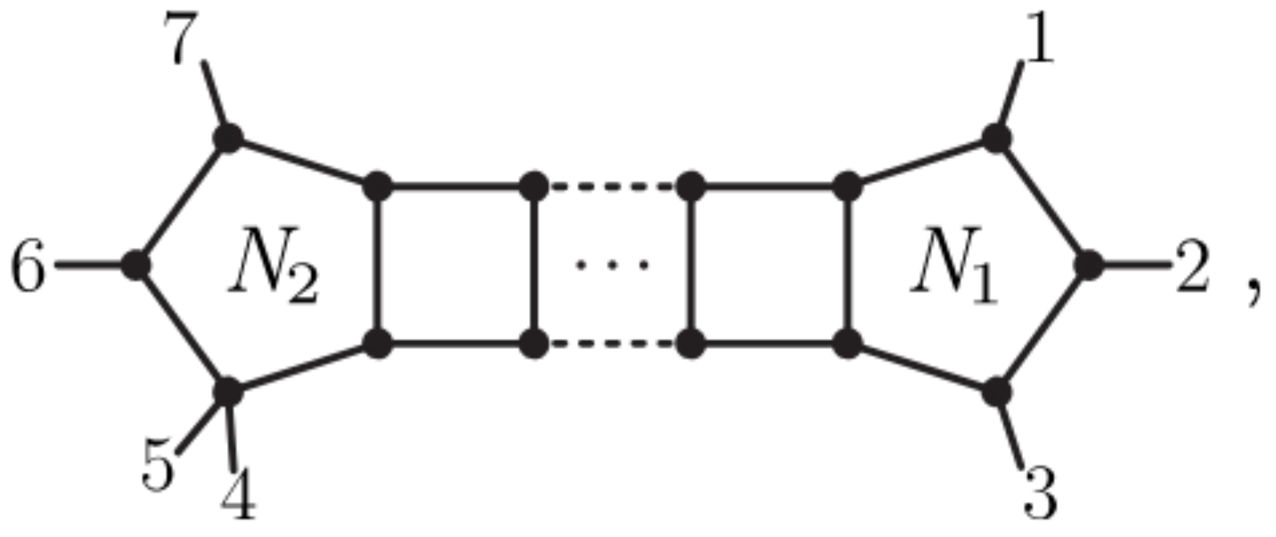


**Rotation of matrix**



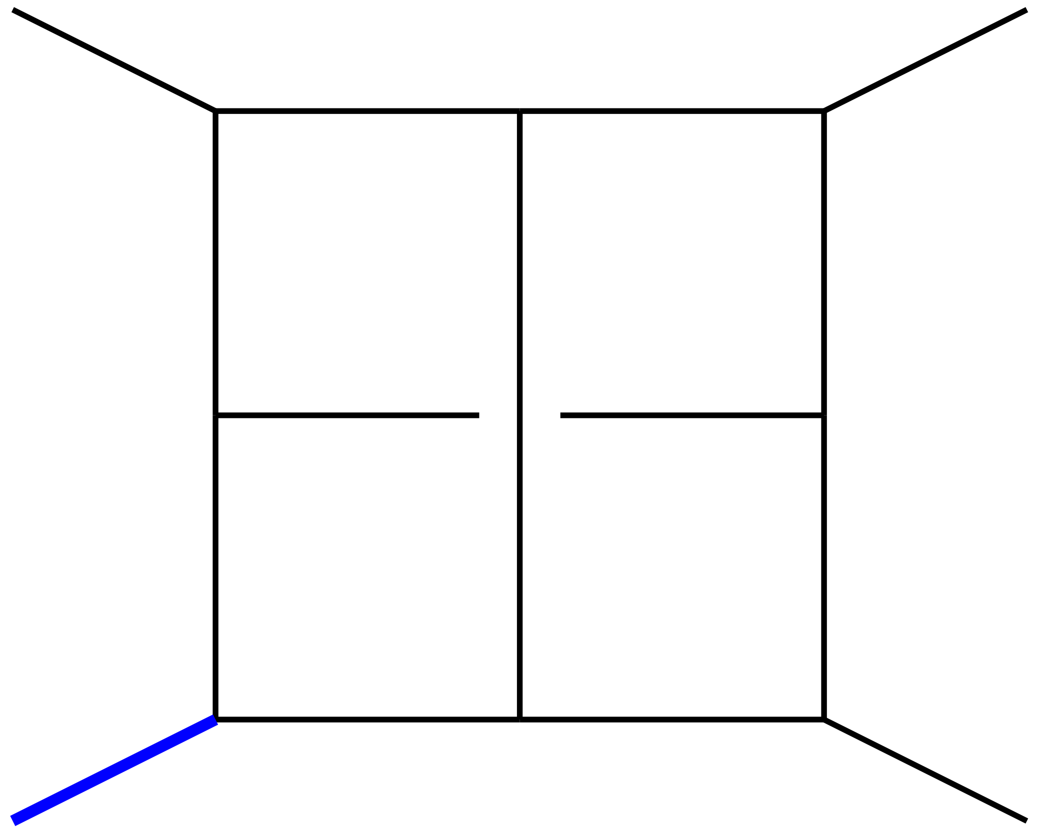

**N=4 sYM**



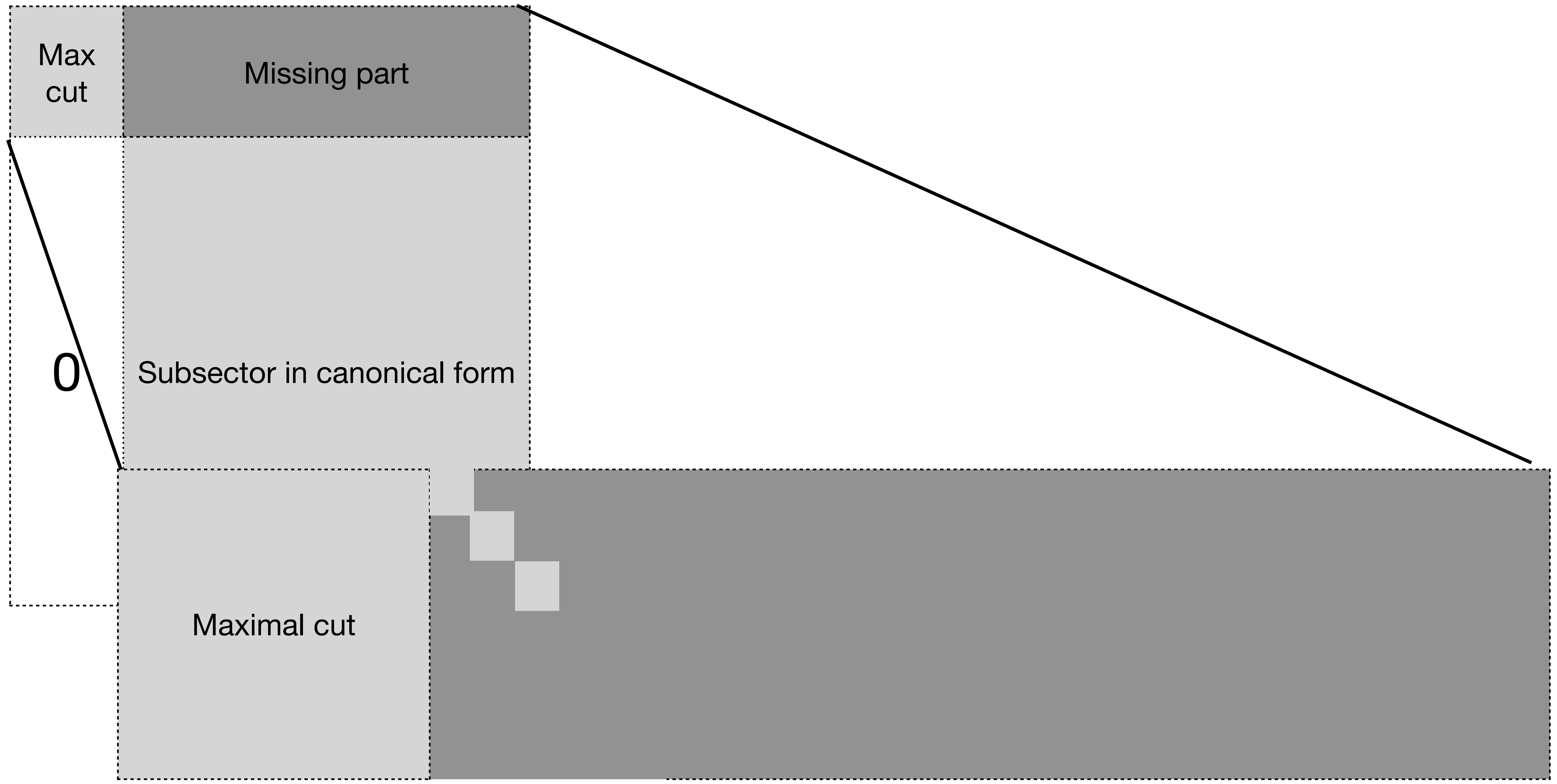



[Bourjaily, McLeod, von Hippel, Wilhelm 2018]

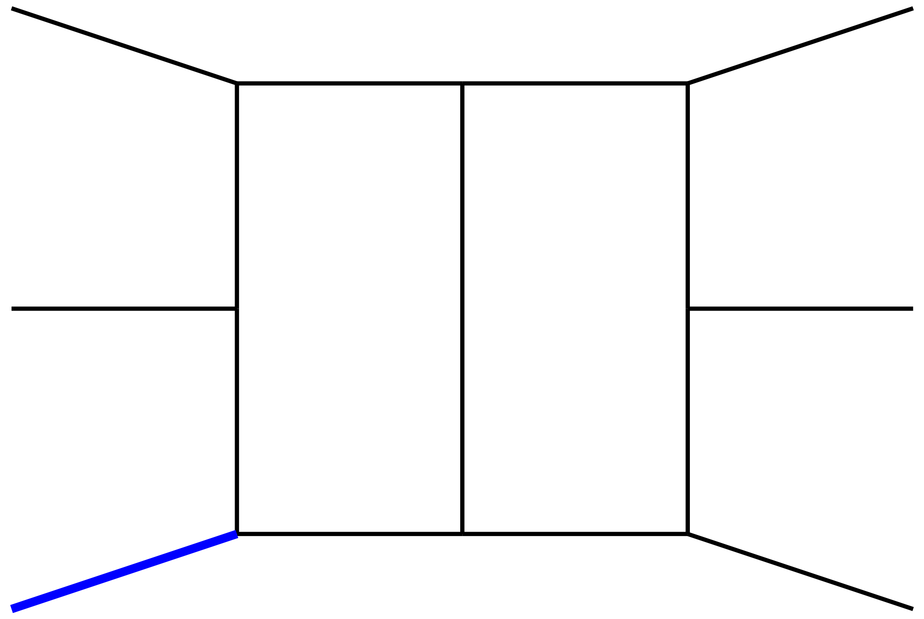
# Finding canonical basis of H

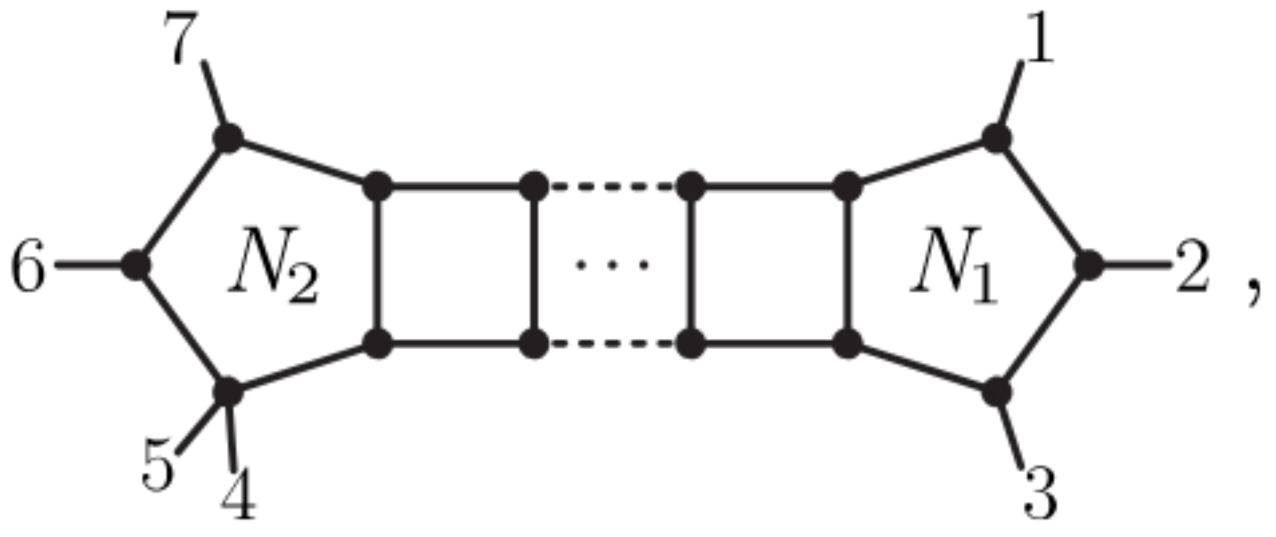


**Rotation of matrix**



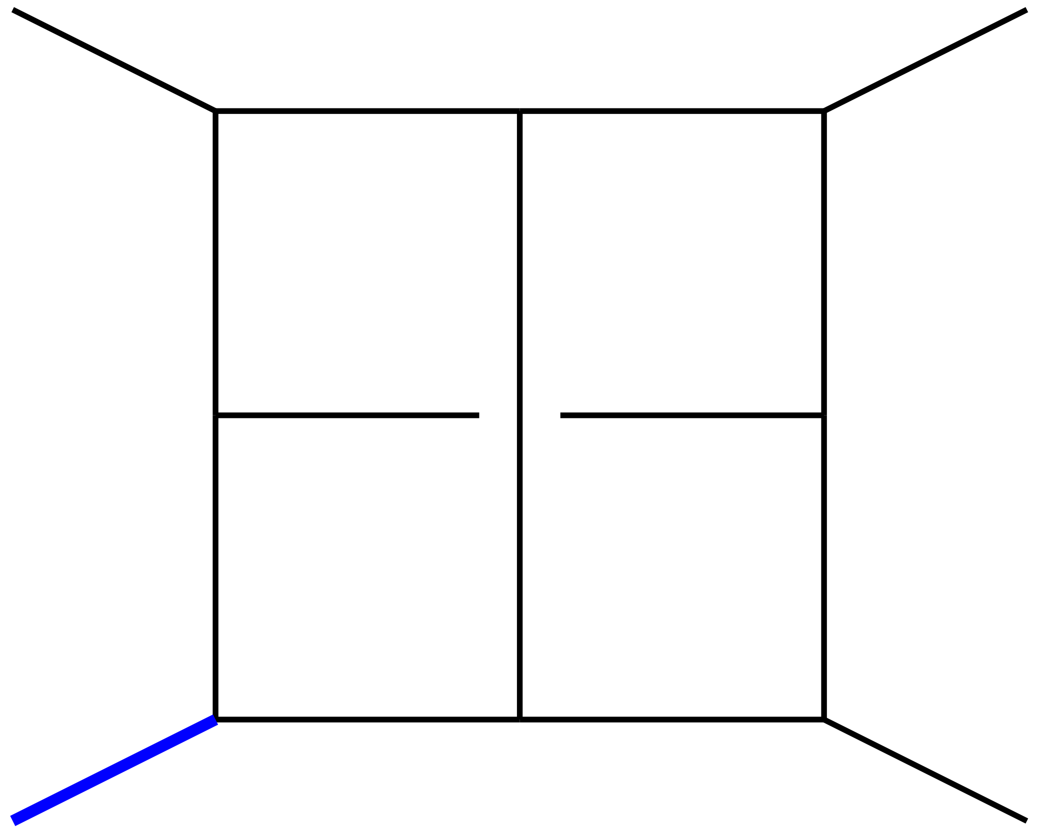

**N=4 sYM**



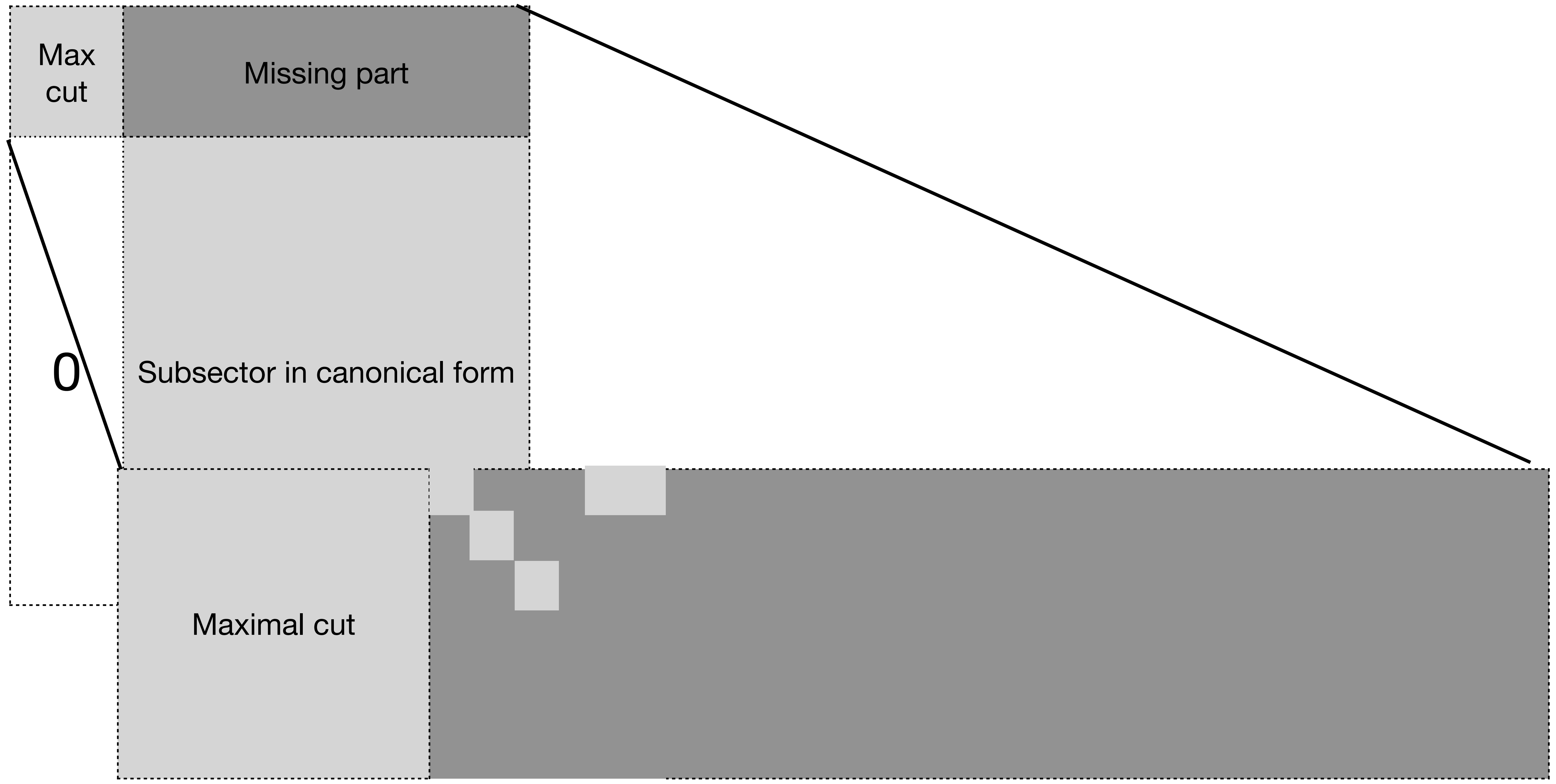



[Bourjaily, McLeod, von Hippel, Wilhelm 2018]

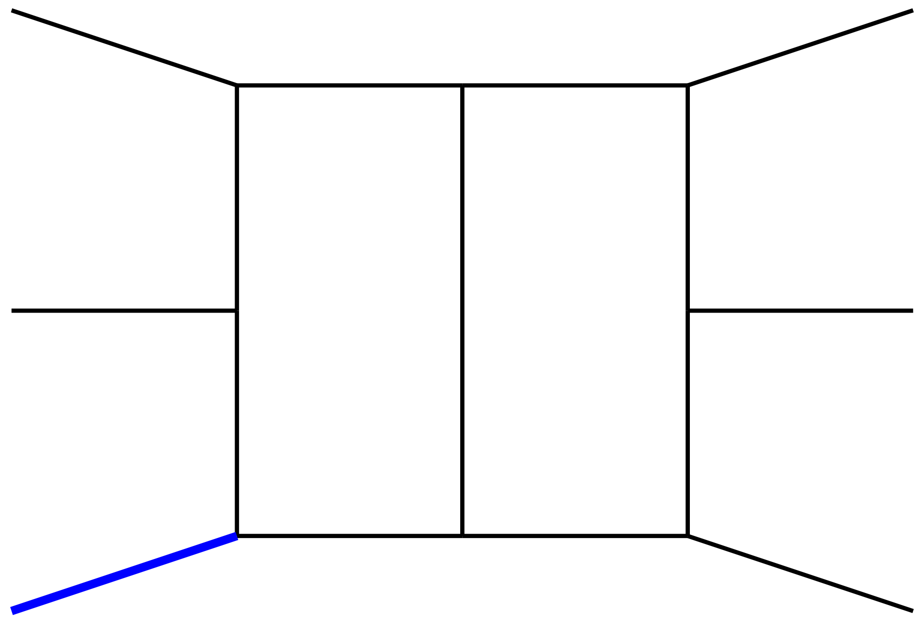
# Finding canonical basis of H

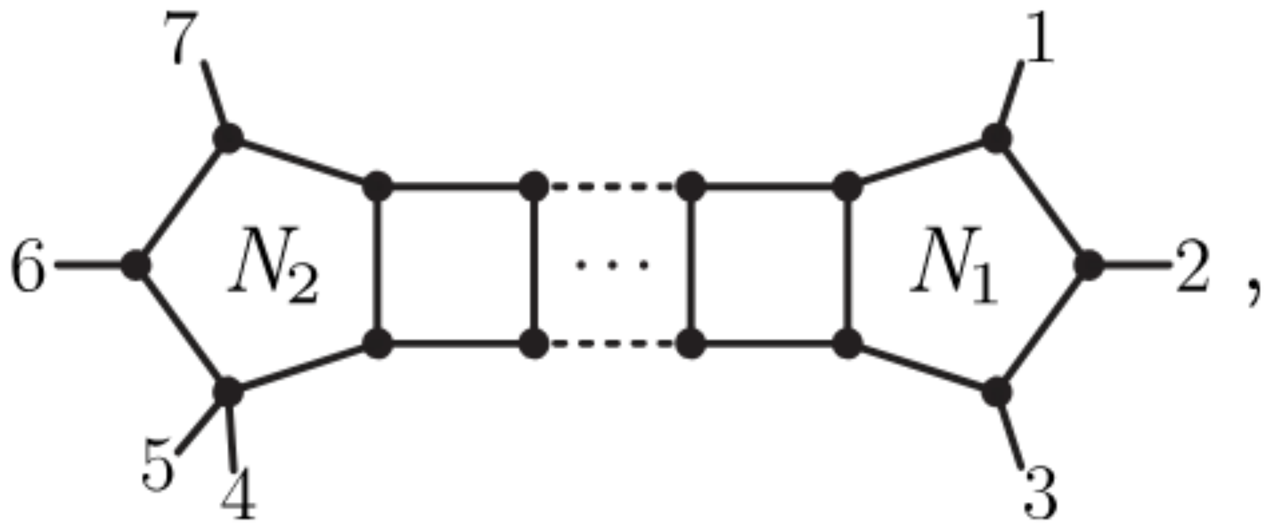


**Rotation of matrix**



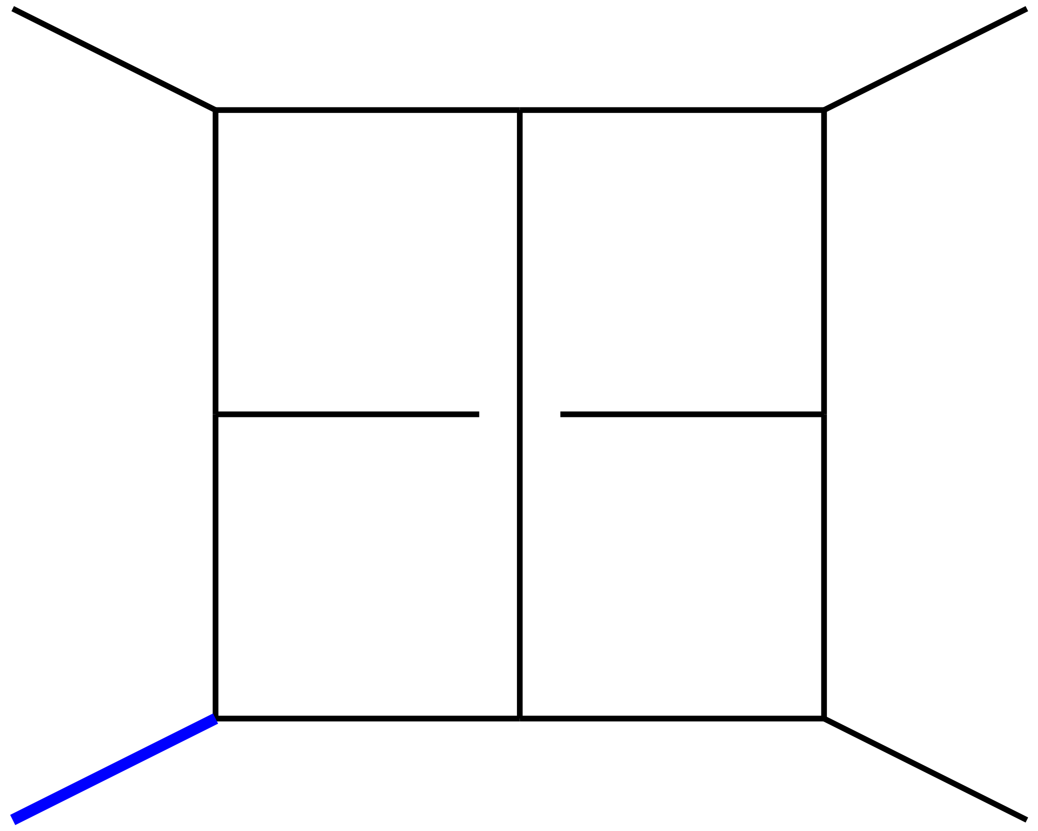

**N=4 sYM**



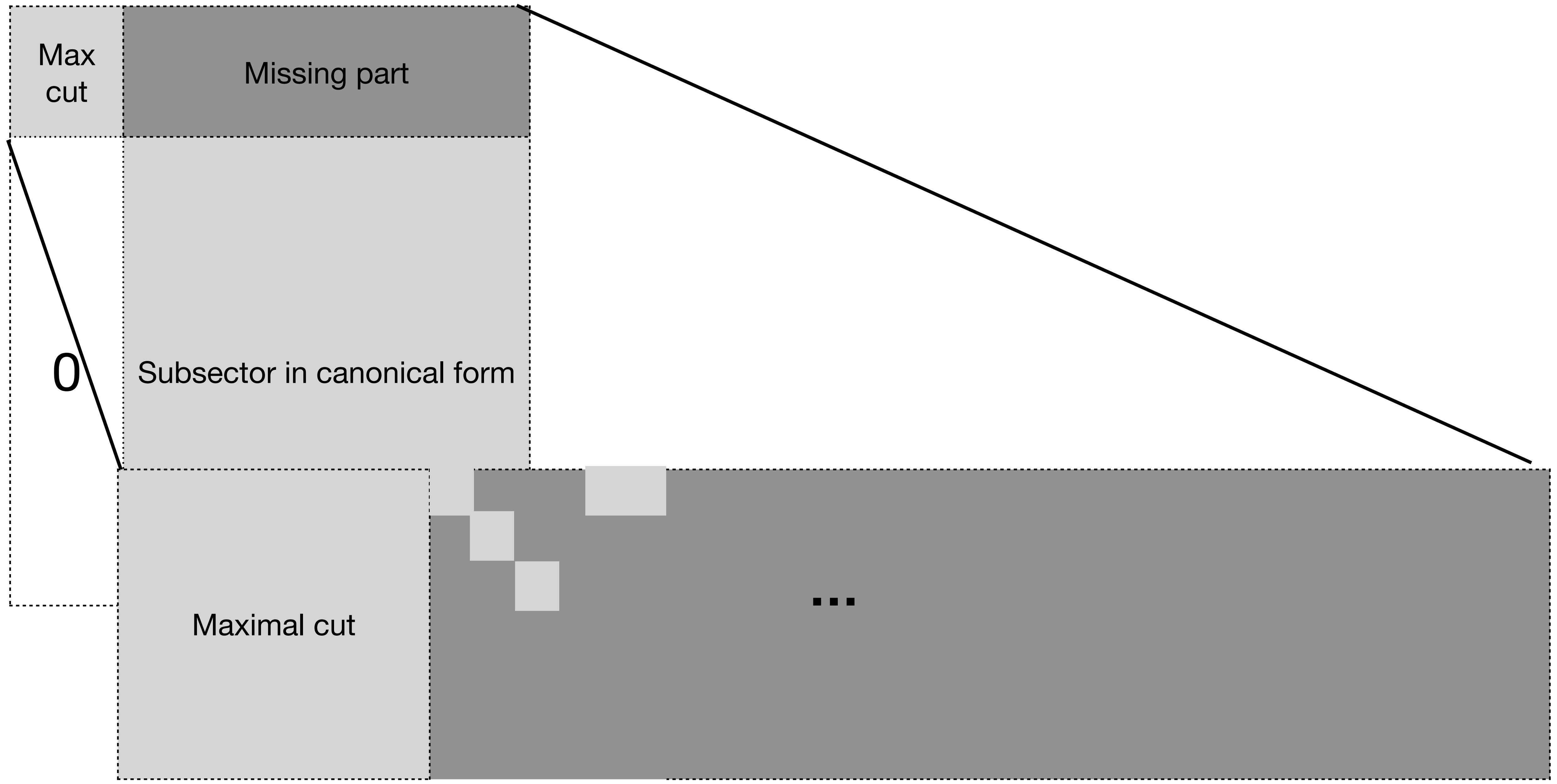



[Bourjaily, McLeod, von Hippel, Wilhelm 2018]

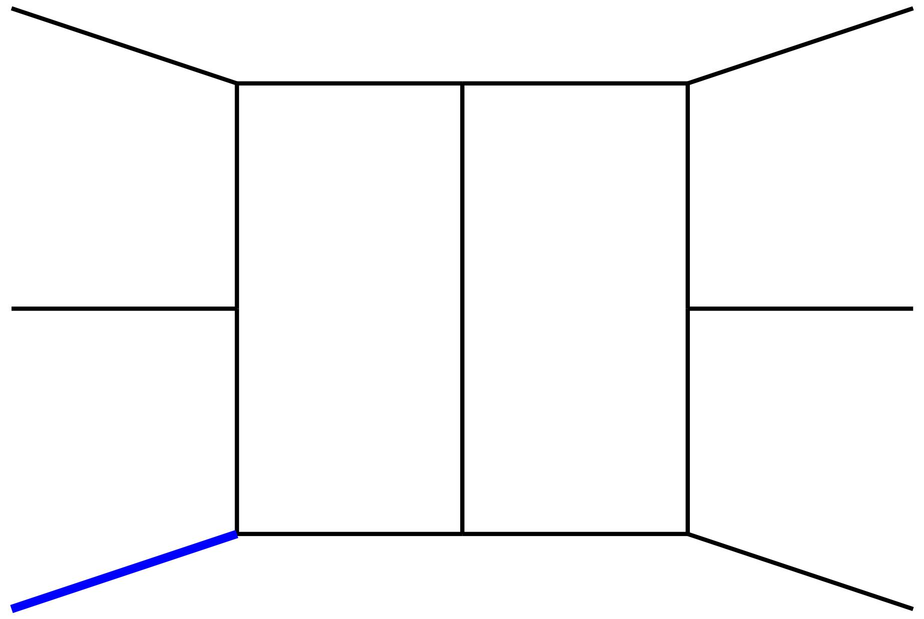
# Finding canonical basis of H

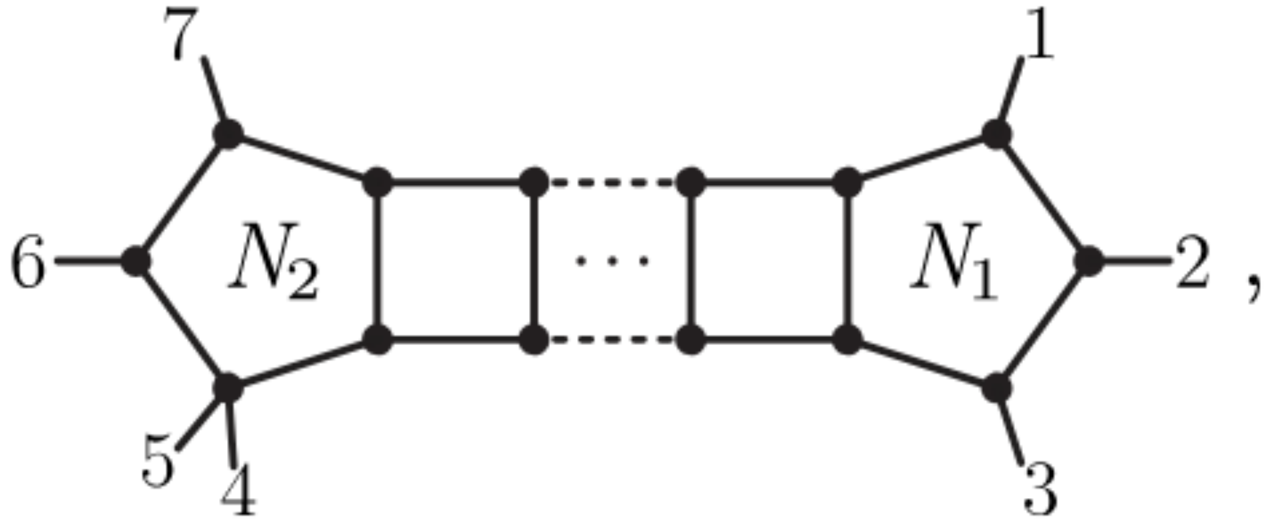


**Rotation of matrix**




**N=4 sYM**






[Bourjaily, McLeod, von Hippel, Wilhelm 2018]

# Summary & Outlook

- We compute three-loop tr$\phi^2$ and tr$\phi^3$ three-point form factor analytically with first-principle method computing Feynman integrals.

- By grading the functions, one can work with more compact basis.

To do :

- Getting full expression of Higgs + jet amplitudes profited from the graded functions.

  **Bottleneck :** numerical IBP reduction off the cut

  Reconstruction of the coefficients (harder than form factor as it's not UT)

- Getting full expression of subleading three-loop three-point tr$\phi^2$ form factor by completing the family H

# Thank You!