

Unitarity Triangle Angles Explained: a Predictive New Quark Mass Matrix Texture

Liverpool Seminar

Paul Harrison
University of Warwick

18th November 2025

JHEP 07 (2025) 155 (arXiv: 2501.18508) with Bill Scott, Rutherford
Appleton Laboratory

Background:

- Mysteries of the quark mass and mixing spectra
- Weak Interaction flavour structure
- SM origins of masses and mixings
- Historical efforts to explain

Background:

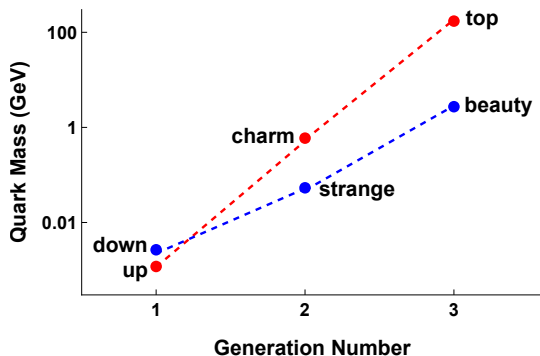
- Mysteries of the quark mass and mixing spectra
- Weak Interaction flavour structure
- SM origins of masses and mixings
- Historical efforts to explain

This work:

- Mysteries of the Unitarity Triangle
- The new mass matrix texture
- Confronting the data
- Symmetries of the texture
- Discussion and conclusions

Mystery of Quark Mass Spectra

- Quark masses show marked hierarchical structure:



- Is quasi-“geometric”:

$$\frac{m_c}{m_t} \simeq 0.0035$$

$$\frac{m_u}{m_c} \simeq 0.0020$$

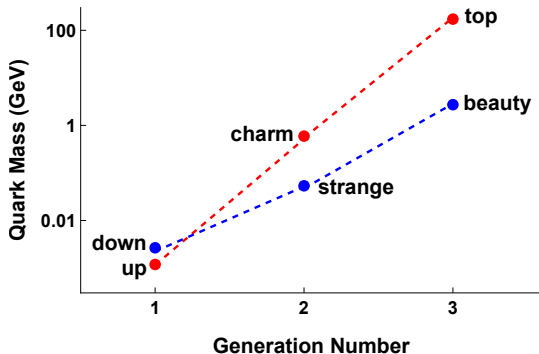
$$\frac{m_s}{m_b} \simeq 0.020$$

$$\frac{m_d}{m_s} \simeq 0.050$$

- Noted by very many authors

Mystery of Quark Mass Spectra

- Quark masses show marked hierarchical structure:



- Is quasi-“geometric”:

$$\frac{m_c}{m_t} \simeq 0.0035$$

$$\frac{m_u}{m_c} \simeq 0.0020$$

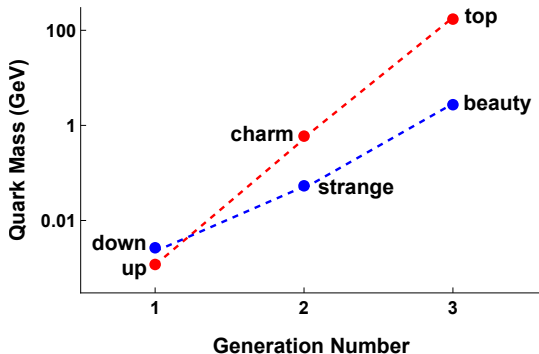
$$\frac{m_s}{m_b} \simeq 0.020$$

$$\frac{m_d}{m_s} \simeq 0.050$$

- Noted by very many authors
- Masses not predicted in the SM
- Hierarchy certainly not explained within SM

Mystery of Quark Mass Spectra

- Quark masses show marked hierarchical structure:



- Is quasi-“geometric”:

$$\frac{m_c}{m_t} \simeq 0.0035$$

$$\frac{m_u}{m_c} \simeq 0.0020$$

$$\frac{m_s}{m_b} \simeq 0.020$$

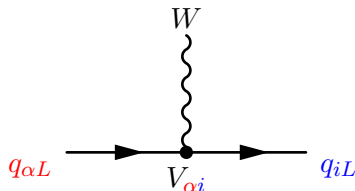
$$\frac{m_d}{m_s} \simeq 0.050$$

- Noted by very many authors
- Masses not predicted in the SM
- Hierarchy certainly not explained within SM
- BSM, Froggatt-Neilsen mechanism has some success

Mystery of Quark Mixing Spectrum

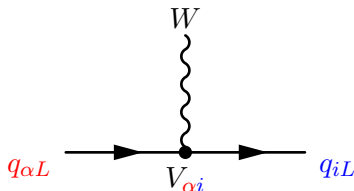
- CKM quark mixing matrix:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



- CKM quark mixing matrix:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



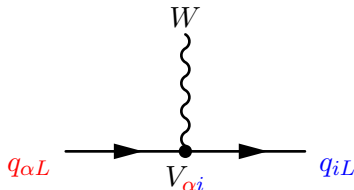
$$\sim \begin{pmatrix} 1 & \lambda & A\lambda^3(\bar{\rho} - i\bar{\eta}) \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3(1 - \bar{\rho} - i\bar{\eta}) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

where $\lambda \equiv |V_{us}| \simeq 0.22$

A , $\bar{\rho}$ and $\bar{\eta} \lesssim \mathcal{O}(1)$

- CKM quark mixing matrix:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



$$\sim \begin{pmatrix} 1 & \lambda & A\lambda^3(\bar{\rho} - i\bar{\eta}) \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3(1 - \bar{\rho} - i\bar{\eta}) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

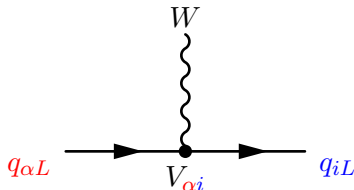
where $\lambda \equiv |V_{us}| \simeq 0.22$

A , $\bar{\rho}$ and $\bar{\eta} \lesssim \mathcal{O}(1)$

- Elements not predicted by the SM
- Strong hierarchy certainly not explained within SM

- CKM quark mixing matrix:

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



$$\sim \begin{pmatrix} 1 & \lambda & A\lambda^3(\bar{\rho} - i\bar{\eta}) \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3(1 - \bar{\rho} - i\bar{\eta}) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

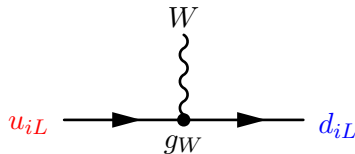
where $\lambda \equiv |V_{us}| \simeq 0.22$

A , $\bar{\rho}$ and $\bar{\eta} \lesssim \mathcal{O}(1)$

- Elements not predicted by the SM
- Strong hierarchy certainly not explained within SM
- But masses and mixings both arise in the Yukawa/Mass matrices

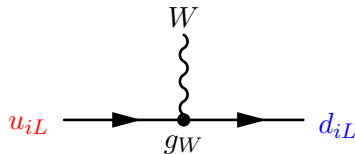
- In the **gauge theory**
- 3 generations of quarks:

$$\begin{pmatrix} u_1 \\ d_1 \end{pmatrix}_L \quad \begin{pmatrix} u_2 \\ d_2 \end{pmatrix}_L \quad \begin{pmatrix} u_3 \\ d_3 \end{pmatrix}_L$$



- In the **gauge theory**
- 3 generations of quarks:

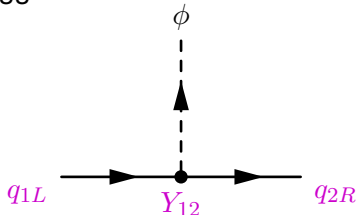
$$\begin{pmatrix} u_1 \\ d_1 \end{pmatrix}_L \quad \begin{pmatrix} u_2 \\ d_2 \end{pmatrix}_L \quad \begin{pmatrix} u_3 \\ d_3 \end{pmatrix}_L$$



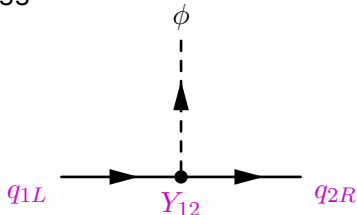
- Write $\underline{u}_w = (u_1, u_2, u_3)^T$ and $\underline{d}_w = (d_1, d_2, d_3)^T$
- $W^\pm$ couplings initially **flavour-diagonal**:

$$\mathcal{L}_W \sim g_W \bar{\underline{u}}_{wL} \cdot \underline{d}_{wL} W^+ + H.C.$$

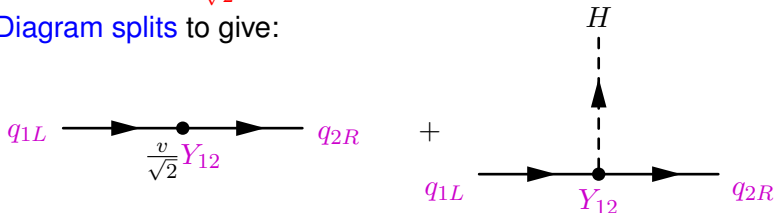
- Fermion **Masses and Mixings** have **common origin** in (Yukawa) couplings of the Higgs to fermions



- Fermion **Masses and Mixings** have **common origin** in (Yukawa) couplings of the Higgs to fermions



- After SSB, $\phi \rightarrow \frac{v}{\sqrt{2}} + H$
- Diagram splits** to give:



- Recall

$$\underline{u}_w = (u_1, u_2, u_3)^T \quad \text{and} \quad \underline{d}_w = (d_1, d_2, d_3)^T$$

- After SSB, Lagrangian for the quark masses is (dropping L/R labels):

$$\mathcal{L}_{Mass} \sim \frac{v}{\sqrt{2}} \bar{\underline{u}}_w \cdot \underline{Y}_u \cdot \underline{u}_w + \frac{v}{\sqrt{2}} \bar{\underline{d}}_w \cdot \underline{Y}_d \cdot \underline{d}_w$$

- Identify

$$\frac{v}{\sqrt{2}} \underline{Y}_u \equiv \underline{M}_u \quad \text{and} \quad \frac{v}{\sqrt{2}} \underline{Y}_d \equiv \underline{M}_d$$

as mass matrices

- Recall

$$\underline{u}_w = (u_1, u_2, u_3)^T \quad \text{and} \quad \underline{d}_w = (d_1, d_2, d_3)^T$$

- After SSB, Lagrangian for the quark masses is (dropping L/R labels):

$$\mathcal{L}_{Mass} \sim \frac{v}{\sqrt{2}} \bar{\underline{u}}_w \cdot Y_u \cdot \underline{u}_w + \frac{v}{\sqrt{2}} \bar{\underline{d}}_w \cdot Y_d \cdot \underline{d}_w$$

- Identify

$$\frac{v}{\sqrt{2}} Y_u \equiv M_u \quad \text{and} \quad \frac{v}{\sqrt{2}} Y_d \equiv M_d$$

as mass matrices

- M_u and M_d are clearly not diagonal
- Can choose basis where they are Hermitian without observable consequences.

Physical Particles?

- Identified as **eigenstates** of M_u and M_d
- So, diagonalise to find them:

$$(u, c, t)^T \equiv \underline{u} = U_u \cdot \underline{u_w} \text{ and } (d, s, b)^T \equiv \underline{d} = U_d \cdot \underline{d_w}$$

- Identified as **eigenstates** of M_u and M_d

- So, diagonalise to find them:

$$(u, c, t)^T \equiv \underline{u} = U_u \cdot \underline{u}_w \text{ and } (d, s, b)^T \equiv \underline{d} = U_d \cdot \underline{d}_w$$

- Then (chiral labels dropped):

$$\mathcal{L}_{M+W} = \bar{\underline{u}} \cdot D_u \cdot \underline{u} + \bar{\underline{d}} \cdot D_d \cdot \underline{d} + g_W \bar{\underline{u}} \cdot U_u \cdot U_d^\dagger \cdot \underline{d} W^+ + \dots$$

- Identified as **eigenstates** of M_u and M_d

- So, diagonalise to find them:

$$(u, c, t)^T \equiv \underline{u} = U_u \cdot \underline{u}_w \text{ and } (d, s, b)^T \equiv \underline{d} = U_d \cdot \underline{d}_w$$

- Then (chiral labels dropped):

$$\mathcal{L}_{M+W} = \bar{\underline{u}} \cdot D_u \cdot \underline{u} + \bar{\underline{d}} \cdot D_d \cdot \underline{d} + g_W \bar{\underline{u}} \cdot U_u \cdot U_d^\dagger \cdot \underline{d} W^+ + \dots$$

- where

$$D_u \equiv U_u \cdot M_u \cdot U_u^\dagger = \text{diag}(m_u, m_c, m_t)$$

$$D_d \equiv U_d \cdot M_d \cdot U_d^\dagger = \text{diag}(m_d, m_s, m_b)$$

$$\text{and } V_{CKM} \equiv U_u \cdot U_d^\dagger \text{ is unitary.}$$

- Thus **masses** and **mixings** both originate in the MMs

- So, can mass ratios and mixings be related?
- One way is with “texture zeroes” - pioneering idea by Harald Fritzsch (1976-78)
- E.g. $M_d \equiv M_F(m_b, a_d, b_d)$

$$= m_b \begin{pmatrix} 0 & a_d & 0 \\ a_d^* & 0 & b_d \\ 0 & b_d^* & 1 \end{pmatrix} : \text{diagonalise} \Rightarrow \begin{aligned} |a_d| &= \sqrt{\frac{m_d}{m_b} \frac{m_s}{m_b}} \sim 0.0044 \\ |b_d| &= \sqrt{\frac{m_s}{m_b}} \sim 0.14 \end{aligned}$$

- So, **can mass ratios and mixings be related?**
- One way is with “texture zeroes” - pioneering idea by **Harald Fritzsch** (1976-78)
- E.g. $M_d \equiv M_F(m_b, a_d, b_d)$

$$= m_b \begin{pmatrix} 0 & a_d & 0 \\ a_d^* & 0 & b_d \\ 0 & b_d^* & 1 \end{pmatrix} : \text{diagonalise} \Rightarrow \begin{aligned} |a_d| &= \sqrt{\frac{m_d}{m_b} \frac{m_s}{m_b}} \sim 0.0044 \\ |b_d| &= \sqrt{\frac{m_s}{m_b}} \sim 0.14 \end{aligned}$$

- Diagonalised by:

$$U_d^\dagger \sim \begin{pmatrix} 1 & s_1^d & s_1^d s_2^d \\ -s_1^d & 1 & s_2^d \\ 0 & -s_2^d & 1 \end{pmatrix} \text{ with } \begin{aligned} s_1^d &\equiv \sin \theta_{12}^d \simeq \sqrt{\frac{m_d}{m_s}} \sim 0.224 \\ s_2^d &\equiv \sin \theta_{23}^d \simeq \sqrt{\frac{m_s}{m_b}} \sim 0.14 \end{aligned}$$

- Already somewhat encouraging.

- BUT s_2^d, s_3^d too big, AND should treat M_u and M_d alike
- Do by writing $M_u = M_F(m_t, a_u, b_u)$ [NB. 8 params for 10 obs ✓]
- Since

$$V_{CKM} = U_u U_d^\dagger (\dagger \Rightarrow \text{inverse for unitary matrix}),$$

is like rotation and a rotation back

- BUT s_2^d, s_3^d too big, AND should treat M_u and M_d alike
- Do by writing $M_u = M_F(m_t, a_u, b_u)$ [NB. 8 params for 10 obs ✓]
- Since

$$V_{CKM} = U_u U_d^\dagger (\dagger \Rightarrow \text{inverse for unitary matrix}),$$

is like rotation and a rotation back

- In complex case, phase enters (gives CP -violation):

$$\tilde{\delta} \equiv \arg(a_u) - \arg(a_d)$$

- Together give:

$$\begin{aligned} V_{us} &\sim \lambda = |s_1^d - s_1^u e^{i\tilde{\delta}}| \\ &= \left| \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} e^{i\tilde{\delta}} \right| \end{aligned}$$

- BUT s_2^d , s_3^d too big, AND should treat M_u and M_d alike
- Do by writing $M_u = M_F(m_t, a_u, b_u)$ [NB. 8 params for 10 obs ✓]
- Since

$$V_{CKM} = U_u U_d^\dagger (\dagger \Rightarrow \text{inverse for unitary matrix}),$$

is like rotation and a rotation back

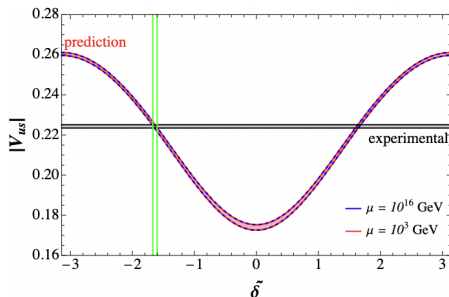
- In complex case, phase enters (gives CP -violation):

$$\tilde{\delta} \equiv \arg(a_u) - \arg(a_d)$$

- Together give:

$$\begin{aligned} V_{us} &\sim \lambda = |s_1^d - s_1^u e^{i\tilde{\delta}}| \\ &= \left| \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} e^{i\tilde{\delta}} \right| \end{aligned}$$

- Good fit ✓



Predictions for V_{cb} and V_{ub}

- Here, another phase enters:

$$\bar{\beta} \equiv \arg(b_u) - \arg(b_d)$$

- One finds:

$$\begin{aligned} V_{cb} &\sim A\lambda^2 = |s_2^d - s_2^u e^{i\bar{\beta}}| \\ &= \left| \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} e^{i\bar{\beta}} \right| \end{aligned}$$

Predictions for V_{cb} and V_{ub}

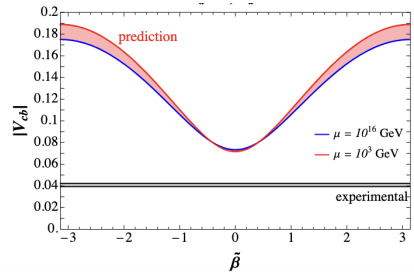
- Here, another phase enters:

$$\bar{\beta} \equiv \arg(b_u) - \arg(b_d)$$

- One finds:

$$\begin{aligned} V_{cb} &\sim A\lambda^2 = |s_2^d - s_2^u e^{i\bar{\beta}}| \\ &= \left| \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} e^{i\bar{\beta}} \right| \end{aligned}$$

Too big: excluded $\times$



Predictions for V_{cb} and V_{ub}

- Here, another phase enters:

$$\bar{\beta} \equiv \arg(b_u) - \arg(b_d)$$

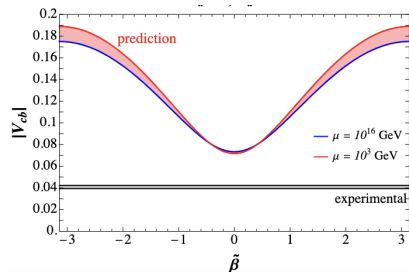
- One finds:

$$\begin{aligned} V_{cb} &\sim A\lambda^2 = |s_2^d - s_2^u e^{i\bar{\beta}}| \\ &= \left| \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} e^{i\bar{\beta}} \right| \end{aligned}$$

Too big: excluded $\times$

- and:

$$\left| \frac{V_{ub}}{V_{cb}} \right| \sim \lambda \simeq \sqrt{\frac{m_u}{m_c}}$$



Predictions for V_{cb} and V_{ub}

- Here, another phase enters:

$$\bar{\beta} \equiv \arg(b_u) - \arg(b_d)$$

- One finds:

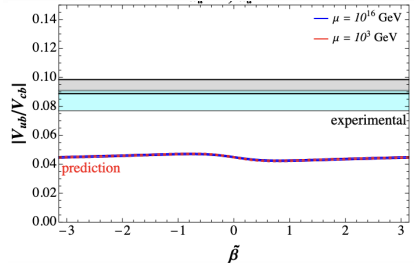
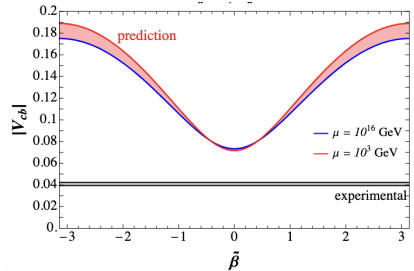
$$\begin{aligned} V_{cb} &\sim A\lambda^2 = |s_2^d - s_2^u e^{i\bar{\beta}}| \\ &= \left| \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} e^{i\bar{\beta}} \right| \end{aligned}$$

Too big: excluded ✗

- and:

$$\left| \frac{V_{ub}}{V_{cb}} \right| \sim \lambda \simeq \sqrt{\frac{m_u}{m_c}}$$

Too small: more excluded ✗



Predictions for V_{cb} and V_{ub}

- Here, another phase enters:

$$\bar{\beta} \equiv \arg(b_u) - \arg(b_d)$$

- One finds:

$$\begin{aligned} V_{cb} &\sim A\lambda^2 = |s_2^d - s_2^u e^{i\bar{\beta}}| \\ &= \left| \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} e^{i\bar{\beta}} \right| \end{aligned}$$

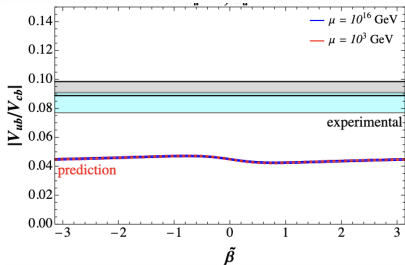
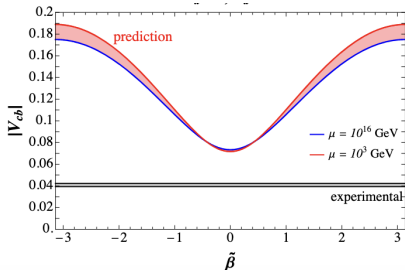
Too big: excluded $\times$

- and:

$$\left| \frac{V_{ub}}{V_{cb}} \right| \sim \lambda \simeq \sqrt{\frac{m_u}{m_c}}$$

Too small: more excluded $\times$

- Figs from **B. Belfatto** and **Z. Berezghiani**, arXiv: 2305.00069. Recent approach to revive Fritzsch using **non-Hermitian MMs** (but 10 pars $\times$)



The Unitarity Triangle

$$V_{CKM} \equiv U_u U_d^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \text{ is unitary.}$$

- ie. complex dot-product of every pair of columns (or rows) is zero.
E.g.

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$

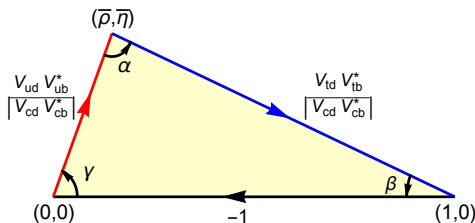
The Unitarity Triangle

$$V_{CKM} \equiv U_u U_d^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \text{ is unitary.}$$

- ie. complex dot-product of every pair of columns (or rows) is zero.
E.g.

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$

- $\Rightarrow$ triangle in complex plane (normalise by $1/|V_{cd}V_{cb}^*|$):



- Base length unity
- 2 parameters, choose:
2 angles or
top vertex = $\bar{\rho} + i\bar{\eta}$
- Area = $\frac{1}{2}\bar{\eta}$
- All CP -violating observables
 $\propto$ Area

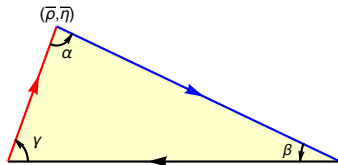
Mysteries of the Unitarity Triangle

- Sides/Angles of UT are **arbitrary in SM**
- But measured angles:

$$\alpha = (91.6 \pm 1.4)^\circ$$

$$\beta = (22.6 \pm 0.4)^\circ$$

$$\gamma = (65.7 \pm 1.3)^\circ$$



consistent with “special” values:

$$(\alpha, \beta, \gamma) \simeq \left(\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}\right) \equiv (\alpha_0, \beta_0, \gamma_0).$$

- Seems striking!

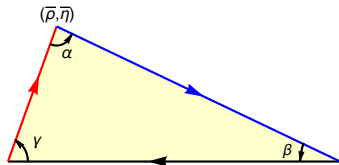
Mysteries of the Unitarity Triangle

- Sides/Angles of UT are **arbitrary in SM**
- But measured angles:

$$\alpha = (91.6 \pm 1.4)^\circ$$

$$\beta = (22.6 \pm 0.4)^\circ$$

$$\gamma = (65.7 \pm 1.3)^\circ$$



consistent with “special” values:

$$(\alpha, \beta, \gamma) \simeq \left(\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}\right) \equiv (\alpha_0, \beta_0, \gamma_0).$$

- Seems striking!
- **Coincidence or smoking gun?**
- $\rightarrow$ Test as clue to what lies behind.

Build Special Angles into a Texture

$$M_q^{HS} \equiv n_q \begin{pmatrix} c_q \lambda_q^4 & b \lambda_q^3 & 0 \\ b \lambda_q^{*3} & b \lambda_q^2 & A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad \begin{array}{l} q = u, d, \\ \lambda_q \text{ complex} \\ \arg(\lambda_q) \text{ unobservable} \end{array}$$

$$(A_0, b, c_u, c_d) \lesssim \mathcal{O}(1)$$

Build Special Angles into a Texture

$$M_q^{HS} \equiv n_q \begin{pmatrix} c_q \lambda_q^4 & b \lambda_q^3 & 0 \\ b \lambda_q^{*3} & b \lambda_q^2 & A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad \begin{array}{l} q = u, d, \\ \lambda_q \text{ complex} \\ \arg(\lambda_q) \text{ unobservable} \end{array}$$

$$(A_0, b, c_u, c_d) \lesssim \mathcal{O}(1)$$

- Complex ratio is *fixed constant*:

$$\frac{\lambda_u}{\lambda_d} \equiv -i \tan \frac{\pi}{8}$$

- Controls *angles* of the UT (see later)

Build Special Angles into a Texture

$$M_q^{HS} \equiv n_q \begin{pmatrix} c_q \lambda_q^4 & b \lambda_q^3 & 0 \\ b \lambda_q^{*3} & b \lambda_q^2 & A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad \begin{array}{l} q = u, d, \\ \lambda_q \text{ complex} \\ \arg(\lambda_q) \text{ unobservable} \end{array}$$

$$(A_0, b, c_u, c_d) \lesssim \mathcal{O}(1)$$

- Complex ratio is *fixed constant*:

$$\frac{\lambda_u}{\lambda_d} \equiv -i \tan \frac{\pi}{8}$$

- Controls **angles** of the UT (see later)
- $\arg \lambda_u / \lambda_d = -i$, is **sole source** of CP violation

$$M_q^{HS} \equiv n_q \begin{pmatrix} c_q \lambda_q^4 & b \lambda_q^3 & 0 \\ b \lambda_q^{*3} & b \lambda_q^2 & A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad \begin{array}{l} q = u, d, \\ \lambda_q \text{ complex} \\ \arg(\lambda_q) \text{ unobservable} \end{array}$$

$$(A_0, b, c_u, c_d) \lesssim \mathcal{O}(1)$$

- Complex ratio is *fixed constant*:

$$\frac{\lambda_u}{\lambda_d} \equiv -i \tan \frac{\pi}{8}$$

- Controls **angles** of the UT (see later)
- $\arg \lambda_u / \lambda_d = -i$, is **sole source** of CP violation
- $|\lambda_u / \lambda_d| \simeq 0.41$ controls relative strength of “ u ” and “ d ” **mass hierarchies**

$$M_q^{HS} \equiv n_q \begin{pmatrix} c_q \lambda_q^4 & b \lambda_q^3 & 0 \\ b \lambda_q^{*3} & b \lambda_q^2 & A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad \begin{array}{l} q = u, d, \\ \lambda_q \text{ complex} \\ \arg(\lambda_q) \text{ unobservable} \end{array}$$

$$(A_0, b, c_u, c_d) \lesssim \mathcal{O}(1)$$

- Complex ratio is *fixed constant*:

$$\frac{\lambda_u}{\lambda_d} \equiv -i \tan \frac{\pi}{8}$$

- Controls angles of the UT (see later)
- $\arg \lambda_u / \lambda_d = -i$, is sole source of CP violation
- $|\lambda_u / \lambda_d| \simeq 0.41$ controls relative strength of “ u ” and “ d ” mass hierarchies

- Complex sum is fitted parameter close to λ :

$$|\lambda_d + \lambda_u| \equiv \lambda_0 = \lambda + \mathcal{O}(\lambda^3).$$

$$M_q^{HS} \equiv n_q \begin{pmatrix} c_q \lambda_q^4 & b \lambda_q^3 & 0 \\ b \lambda_q^{*3} & b \lambda_q^2 & A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad \begin{array}{l} q = u, d, \\ \lambda_q \text{ complex} \\ \arg(\lambda_q) \text{ unobservable} \end{array}$$

$$(A_0, b, c_u, c_d) \lesssim \mathcal{O}(1)$$

- Complex ratio is *fixed constant*:

$$\frac{\lambda_u}{\lambda_d} \equiv -i \tan \frac{\pi}{8}$$

- Controls angles of the UT (see later)
- $\arg \lambda_u / \lambda_d = -i$, is sole source of CP violation
- $|\lambda_u / \lambda_d| \simeq 0.41$ controls relative strength of “ u ” and “ d ” mass hierarchies

- Complex sum is fitted parameter close to λ :

$$|\lambda_d + \lambda_u| \equiv \lambda_0 = \lambda + \mathcal{O}(\lambda^3).$$

- Describes 10 observables with 7 real parameters

- Diagonalise $\rightarrow$ masses:

$$D_q = U_q M_q^{HS} U_q^\dagger = m_3^q \begin{pmatrix} (c_q - b)\lambda_q^4 & 0 & 0 \\ 0 & b\lambda_q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = u, d,$$

- Good for mass hierarchy ($\lambda_u, \lambda_d \ll 1$) ✓

- Diagonalise $\rightarrow$ masses:

$$D_q = U_q M_q^{HS} U_q^\dagger = m_3^q \begin{pmatrix} (c_q - b)\lambda_q^4 & 0 & 0 \\ 0 & b\lambda_q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = u, d,$$

- Good for mass hierarchy ($\lambda_u, \lambda_d \ll 1$) ✓
- 3 free parameters (at LO): b, c_u, c_d (to fit 4 mass ratios)
- $\Rightarrow$ one constraint/prediction (LO):

$$\frac{m_c}{m_t} \frac{m_b}{m_s} = \left| \frac{\lambda_u}{\lambda_d} \right|^2 = \tan^2 \frac{\pi}{8} = \left\{ \begin{matrix} 0.172 \text{ (LO)} \\ 0.176 \text{ (NLO)} \end{matrix} \right\} \text{ c.f. } 0.177 \pm 0.002 \text{ (exp)} \checkmark$$

- Diagonalise $\rightarrow$ masses:

$$D_q = U_q M_q^{HS} U_q^\dagger = m_q^3 \begin{pmatrix} (c_q - b)\lambda_q^4 & 0 & 0 \\ 0 & b\lambda_q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = u, d,$$

- Good for mass hierarchy ($\lambda_u, \lambda_d \ll 1$) ✓
- 3 free parameters (at LO): b, c_u, c_d (to fit 4 mass ratios)
- $\Rightarrow$ one constraint/prediction (LO):

$$\frac{m_c}{m_t} \frac{m_b}{m_s} = \left| \frac{\lambda_u}{\lambda_d} \right|^2 = \tan^2 \frac{\pi}{8} = \left\{ \begin{matrix} 0.172 \text{ (LO)} \\ 0.176 \text{ (NLO)} \end{matrix} \right\} \text{ c.f. } 0.177 \pm 0.002 \text{ (exp)} \checkmark$$

- Fits any m_u, m_d ✓ (no prediction here).

Leading-order Solution (Quark Mixing)

- Diagonalised by 2×2 (complex) rotations in 23 and 12 spaces.
- Small entries induced in the 13 elements of U_q :

$$U_q \simeq \begin{pmatrix} 1 & \pm \lambda_q & A_0 \lambda_q^3 \\ \mp \lambda_q^* & 1 & -A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad q = u, d.$$

Leading-order Solution (Quark Mixing)

- Diagonalised by 2×2 (complex) rotations in 23 and 12 spaces.
- Small entries induced in the 13 elements of U_q :

$$U_q \simeq \begin{pmatrix} 1 & \pm \lambda_q & A_0 \lambda_q^3 \\ \mp \lambda_q^* & 1 & -A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad q = u, d.$$

- Combine U_u and U_d :

$$\Rightarrow V_{CKM} = U_u U_d^\dagger \simeq \begin{pmatrix} 1 & \lambda_0 & A_0 \lambda_0^2 \lambda_u \\ -\lambda_0 & 1 & A_0 \lambda_0^2 \\ A_0 \lambda_0^2 \lambda_d^* & -A_0 \lambda_0^2 & 1 \end{pmatrix}$$

Leading-order Solution (Quark Mixing)

- Diagonalised by 2×2 (complex) rotations in 23 and 12 spaces.
- Small entries induced in the 13 elements of U_q :

$$U_q \simeq \begin{pmatrix} 1 & \pm \lambda_q & A_0 \lambda_q^3 \\ \mp \lambda_q^* & 1 & -A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix}, \quad q = u, d.$$

- Combine U_u and U_d :

$$\Rightarrow V_{CKM} = U_u U_d^\dagger \simeq \begin{pmatrix} 1 & \lambda_0 & A_0 \lambda_0^2 \lambda_u \\ -\lambda_0 & 1 & A_0 \lambda_0^2 \\ A_0 \lambda_0^2 \lambda_d^* & -A_0 \lambda_0^2 & 1 \end{pmatrix}$$

- C.f. Wolfenstein form:

$$V_{CKM} = \begin{pmatrix} 1 & \lambda & A\lambda^3(\bar{\rho} - i\bar{\eta}) \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3(1 - \bar{\rho} - i\bar{\eta}) & -A\lambda^2 & 1 \end{pmatrix} \Rightarrow \begin{cases} \lambda \simeq \lambda_0 \checkmark \\ A \simeq A_0 \checkmark \\ (\bar{\rho} + i\bar{\eta}) \simeq \frac{\lambda_u^*}{\lambda_0} \end{cases}$$

The UT Angles

- Have deduced that:

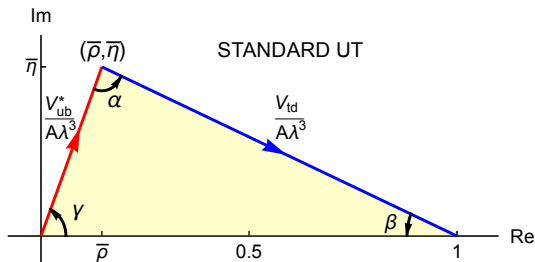
$$\lambda_u^* \simeq \lambda(\bar{\rho} + i\bar{\eta}) = \frac{V_{ub}^*}{A\lambda^2}$$

$$\lambda_d^* \simeq \lambda(1 - \bar{\rho} - i\bar{\eta}) = \frac{V_{td}}{A\lambda^2}$$

$$\Rightarrow \gamma \simeq \arg \lambda_u^*$$

$$\beta \simeq \arg \lambda_d$$

$$\text{and } \alpha \simeq \arg(-\frac{\lambda_u}{\lambda_d})$$



The UT Angles

- Have deduced that:

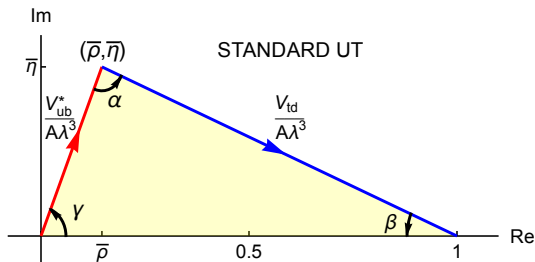
$$\lambda_u^* \simeq \lambda(\bar{\rho} + i\bar{\eta}) = \frac{V_{ub}^*}{A\lambda^2}$$

$$\lambda_d^* \simeq \lambda(1 - \bar{\rho} - i\bar{\eta}) = \frac{V_{td}}{A\lambda^2}$$

$$\Rightarrow \gamma \simeq \arg \lambda_u^*$$

$$\beta \simeq \arg \lambda_d$$

$$\text{and } \alpha \simeq \arg(-\frac{\lambda_u}{\lambda_d})$$



- Recall, HS texture asserts $\frac{\lambda_u}{\lambda_d} = -i \tan \frac{\pi}{8}$

► $\Rightarrow \alpha \simeq \frac{\pi}{2}$ ✓

► $\Rightarrow \tan \beta = |\frac{\lambda_u}{\lambda_d}|$ (see Figure).

► $\Rightarrow \beta \simeq \frac{\pi}{8}$ ✓

- Data from PDG
- Renormalise to common scale ($\mu = m_t$)
- Fit using full numerical diagonalisation

- Data from **PDG**
- Renormalise to common scale ($\mu = m_t$)
- Fit using full numerical diagonalisation
- $\rightarrow$ poor fit: $\chi^2/dof \simeq 100/3!$
- Tension between fitted values of A , m_c/m_t and m_s/m_b .
- **Disaster?**

- Data from **PDG**
- Renormalise to common scale ($\mu = m_t$)
- Fit using full numerical diagonalisation
- $\rightarrow$ poor fit: $\chi^2/dof \simeq 100/3!$
- Tension between fitted values of A , m_c/m_t and m_s/m_b .
- **Disaster?**
- **Not necessarily!**

- Data from **PDG**
- Renormalise to common scale ($\mu = m_t$)
- Fit using full numerical diagonalisation
- $\rightarrow$ poor fit: $\chi^2/dof \simeq 100/3!$
- Tension between fitted values of A , m_c/m_t and m_s/m_b .
- **Disaster?**
- **Not necessarily!**
- Because these quantities “run” with renormalisation scale
- $\sim 13\%$ from weak to GUT scales: $A(\uparrow)$, $m_c/m_t(\uparrow)$ and $m_s/m_b(\downarrow)$.
- λ , α , β , m_u/m_c and m_d/m_s are $\sim$ invariant.
- $\Rightarrow$ vary μ

- Fit $\chi^2/\text{d.o.f} \simeq 1.01/2$
- Best fit renormalisation scale:
 $\mu \sim (0.3 \rightarrow 3) \times 10^4 \text{ TeV}$
- Fitted values of the free parameters:
 - ▶ $\lambda_0 = 0.22646$
 - ▶ $A_0 = 0.854$
 - ▶ $b = 0.462$
 - ▶ $c_u = 0.344$
 - ▶ $c_d = -0.040$

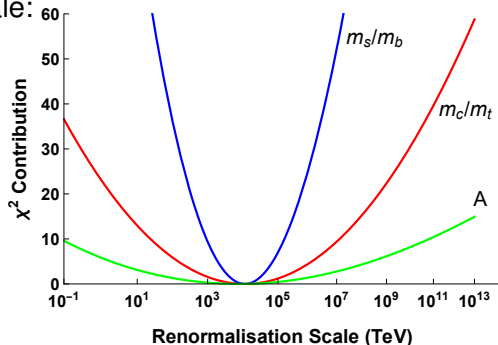
- Fit $\chi^2/\text{d.o.f} \simeq 1.01/2$

- Best fit renormalisation scale:

$$\mu \sim (0.3 \rightarrow 3) \times 10^4 \text{ TeV}$$

- Fitted values of the free parameters:

- ▶ $\lambda_0 = 0.22646$
- ▶ $A_0 = 0.854$
- ▶ $b = 0.462$
- ▶ $c_u = 0.344$
- ▶ $c_d = -0.040$



- Three curves minimise at
common scale $\sim 10^4 \text{ TeV}$

Observable	Input Renormali- sed to $\mu = 10^4$ TeV	Fitted Value at $\mu = 10^4$ TeV
$ m_u/m_c (\times 10^3)$	2.00 ± 0.05	2.00
$ m_d/m_s (\times 10^2)$	4.97 ± 0.06	4.97
$m_c/m_t (\times 10^3)$	3.46 ± 0.03	3.46
$m_s/m_b (\times 10^2)$	1.968 ± 0.008	1.968
λ	0.2250 ± 0.0007	0.2250
A	0.88 ± 0.02	0.88
$\bar{\rho}$	0.159 ± 0.009	0.152
$\bar{\eta}$	0.352 ± 0.007	0.348
UT Angles	—	Prediction from Fit
$\alpha (^\circ)$	91.6 ± 1.4	91.30 ± 0.02
$\beta (^\circ)$	22.6 ± 0.4	22.3 ± 0.1
$\gamma (^\circ)$	65.7 ± 1.3	66.4 ± 0.1

Fitted values in table are predictions

The Leading Order UT (LO-UT)

- Define useful complex constants:

$$z_0 \equiv \lambda_u^* / \lambda_0 = i s_0 e^{-i\beta_0} = \rho_0 + i\eta_0,$$

$$\bar{z}_0 \equiv \lambda_d^* / \lambda_0 = c_0 e^{-i\beta_0} = 1 - z_0,$$

where

$$s_0 \equiv \sin \beta_0; \quad c_0 \equiv \cos \beta_0; \quad \eta_0 = s_0 c_0 = \frac{1}{2\sqrt{2}} \quad \text{and} \quad \rho_0 = s_0^2.$$

The Leading Order UT (LO-UT)

- Define useful complex constants:

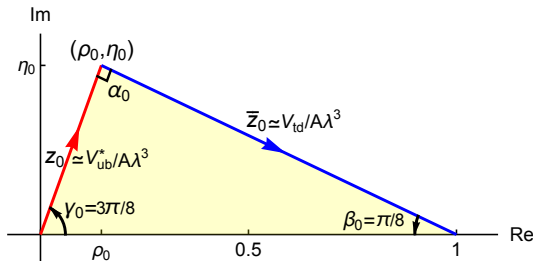
$$z_0 \equiv \lambda_u^* / \lambda_0 = i s_0 e^{-i\beta_0} = \rho_0 + i\eta_0,$$

$$\bar{z}_0 \equiv \lambda_d^* / \lambda_0 = c_0 e^{-i\beta_0} = 1 - z_0,$$

where

$$s_0 \equiv \sin \beta_0; \quad c_0 \equiv \cos \beta_0; \quad \eta_0 = s_0 c_0 = \frac{1}{2\sqrt{2}} \quad \text{and} \quad \rho_0 = s_0^2.$$

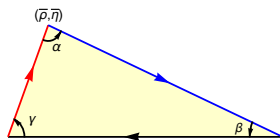
- Use to construct LO-UT



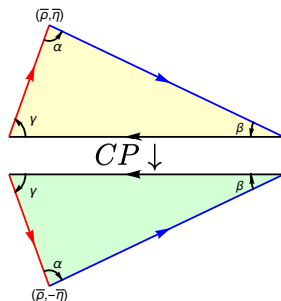
- Properties of the *paired system* (M_u , M_d), rather than of either in isolation
- Could be viewed as consequence of forms, or, preferably, as ab initio symmetries which constrain (M_u , M_d) forms
- Outlined below

CP Transformation and Rephasing

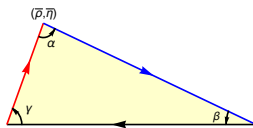
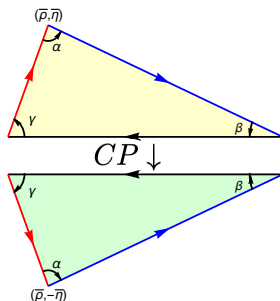
- CP :
- Under CP , all complex numbers in the MMs are **complex-conjugated**



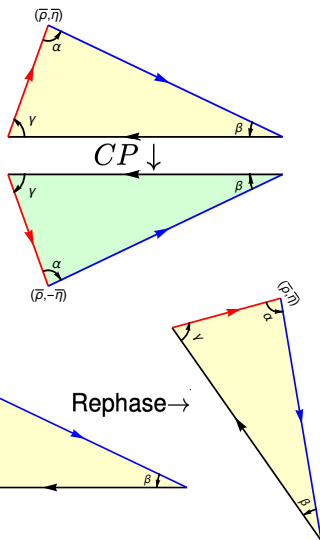
- CP :
- Under CP , all complex numbers in the MMs are **complex-conjugated**
- Observable effect is to **flip orientation** of UT in complex plane ($\bar{\eta} \rightarrow -\bar{\eta}$)
- Unless $\bar{\eta} = 0$ (CP is conserved)



- CP :
- Under CP , all complex numbers in the MMs are **complex-conjugated**
- Observable effect is to **flip orientation** of UT in complex plane ($\bar{\eta} \rightarrow -\bar{\eta}$)
- Unless $\bar{\eta} = 0$ (CP is conserved)
- Rephasing:
- Simultaneous **phase changes** of M_u and M_d unobservable

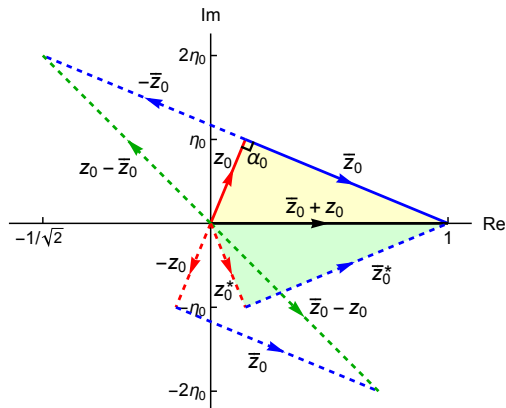


- CP :
- Under CP , all complex numbers in the MMs are **complex-conjugated**
- Observable effect is to **flip orientation** of UT in complex plane ($\bar{\eta} \rightarrow -\bar{\eta}$)
- Unless $\bar{\eta} = 0$ (CP is conserved)
- Rephasing:
- Simultaneous **phase changes** of M_u and M_d unobservable
- UT simply **rotates** in complex plane
- (Physical) **shape and size invariant**



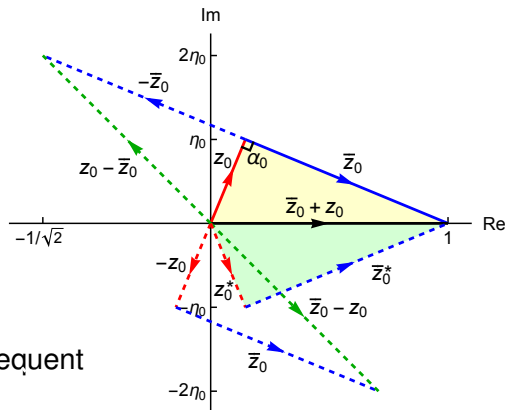
Symmetry for $\alpha_0 = \frac{\pi}{2}$

- In HS texture, simple sign change of z_0 (or of $\bar{z}_0$, but not both), **flips orientation** of the UT (see fig $\rightarrow$)
- Is **only** observable effect
- But iff $\alpha = \pm \frac{\pi}{2}$



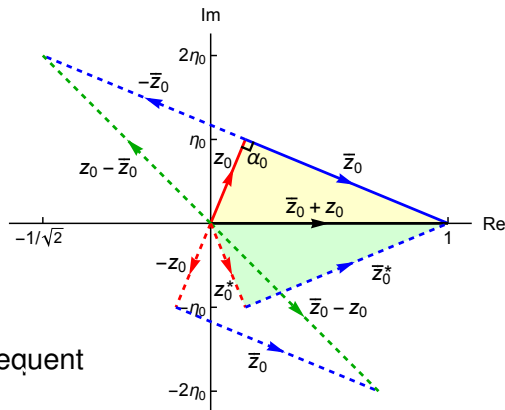
Symmetry for $\alpha_0 = \frac{\pi}{2}$

- In HS texture, simple sign change of z_0 (or of $\bar{z}_0$, but not both), **flips orientation** of the UT (see fig $\rightarrow$)
- Is **only** observable effect
- But iff $\alpha = \pm \frac{\pi}{2}$
- **Equivalent to CP** transformation
- **Can be reversed** by a subsequent **actual CP** transformation



Symmetry for $\alpha_0 = \frac{\pi}{2}$

- In HS texture, simple sign change of z_0 (or of $\bar{z}_0$, but not both), **flips orientation** of the UT (see fig $\rightarrow$)
- Is **only** observable effect
- But iff $\alpha = \pm \frac{\pi}{2}$
- **Equivalent to CP** transformation
- **Can be reversed** by a subsequent **actual CP** transformation
- Symmetry is **good to all orders**

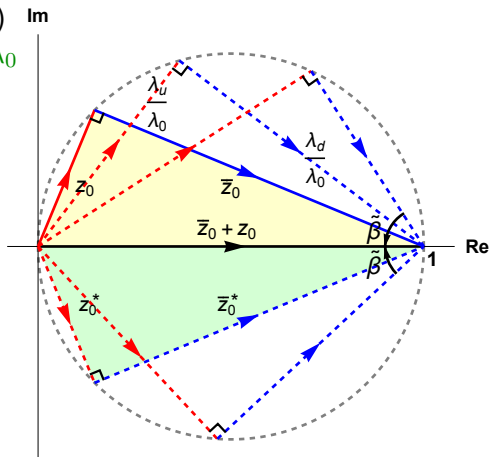


Symmetry for $\beta_0 = \frac{\pi}{8}$

- First consider $\beta_0 = \tilde{\beta} \neq \frac{\pi}{8}$ (fig→) keeping $\alpha = \frac{\pi}{2}$ and $\lambda_u + \lambda_d = \lambda_0$
- Clearly now

$$\left| \frac{\lambda_u}{\lambda_d} \right| = \tan \tilde{\beta},$$

$$\text{and } -\frac{\pi}{2} < \tilde{\beta} < \frac{\pi}{2}$$



Symmetry for $\beta_0 = \frac{\pi}{8}$

- First consider $\beta_0 = \tilde{\beta} \neq \frac{\pi}{8}$ (fig→) keeping $\alpha = \frac{\pi}{2}$ and $\lambda_u + \lambda_d = \lambda_0$
- Clearly now

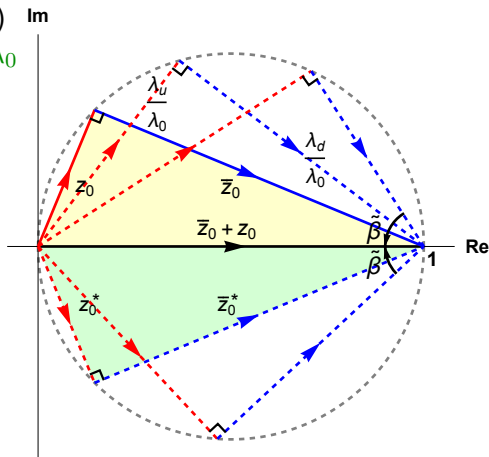
$$\left| \frac{\lambda_u}{\lambda_d} \right| = \tan \tilde{\beta},$$

$$\text{and } -\frac{\pi}{2} < \tilde{\beta} < \frac{\pi}{2}$$

- Consider the following rotation of λ_d :

$$\tilde{\beta} \rightarrow \tilde{\beta} - \frac{\pi}{4} \quad (*)$$

- Iff $\tilde{\beta} = \frac{\pi}{8}$, the result is just a *CP* transformation



Symmetry for $\beta_0 = \frac{\pi}{8}$

- First consider $\beta_0 = \tilde{\beta} \neq \frac{\pi}{8}$ (fig→) keeping $\alpha = \frac{\pi}{2}$ and $\lambda_u + \lambda_d = \lambda_0$
- Clearly now

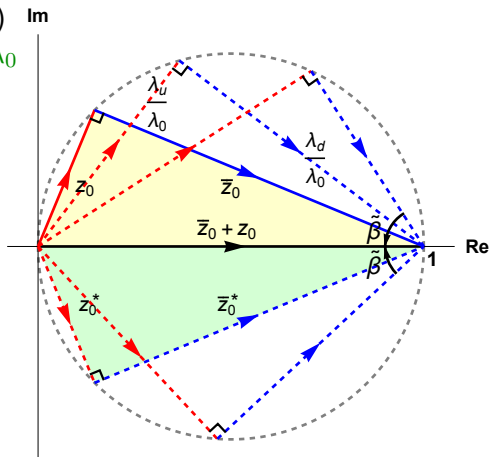
$$\left| \frac{\lambda_u}{\lambda_d} \right| = \tan \tilde{\beta},$$

$$\text{and } -\frac{\pi}{2} < \tilde{\beta} < \frac{\pi}{2}$$

- Consider the following rotation of λ_d :

$$\tilde{\beta} \rightarrow \tilde{\beta} - \frac{\pi}{4} \quad (*)$$

- Iff $\tilde{\beta} = \frac{\pi}{8}$, the result is just a *CP* transformation
- $\Rightarrow$ to fix $\beta_0 = \frac{\pi}{8}$ require symmetry under transformation $(*)$ followed by *CP* flip



- Proposed geometric-hierarchical MM texture
- Mass hierarchy “slopes” are related to UT sides
- Symmetries constrain forms $\rightarrow \alpha \simeq \frac{\pi}{2}$ and $\beta \simeq \frac{\pi}{8}$

- Proposed geometric-hierarchical MM texture
- Mass hierarchy “slopes” are related to UT sides
- Symmetries constrain forms $\rightarrow \alpha \simeq \frac{\pi}{2}$ and $\beta \simeq \frac{\pi}{8}$
- Hierarchy not explained, (but standard model-building methods can achieve that, e.g. F-N Mechanism)

- Proposed geometric-hierarchical MM texture
- Mass hierarchy “slopes” are related to UT sides
- Symmetries constrain forms $\rightarrow \alpha \simeq \frac{\pi}{2}$ and $\beta \simeq \frac{\pi}{8}$
- Hierarchy not explained, (but standard model-building methods can achieve that, e.g. F-N Mechanism)
- M_u and M_d exploit 7 pars to fit 10 observables with $\chi^2/\text{d.o.f} \simeq 1/2$

- Proposed geometric-hierarchical MM texture
- Mass hierarchy “slopes” are related to UT sides
- Symmetries constrain forms $\rightarrow \alpha \simeq \frac{\pi}{2}$ and $\beta \simeq \frac{\pi}{8}$
- Hierarchy not explained, (but standard model-building methods can achieve that, e.g. F-N Mechanism)
- M_u and M_d exploit 7 pars to fit 10 observables with $\chi^2/\text{d.o.f} \simeq 1/2$
- Precise prediction of quark mass double ratio:

$$\frac{m_c}{m_t} \frac{m_b}{m_s} = \left| \frac{\lambda_u}{\lambda_d} \right|^2 = \tan^2 \frac{\pi}{8} (1 + \mathcal{O}(\lambda_0^2)) = 0.176 \pm 0.001$$

c.f. 0.177 ± 0.002 (exp)

- Proposed geometric-hierarchical MM texture
- Mass hierarchy “slopes” are related to UT sides
- Symmetries constrain forms $\rightarrow \alpha \simeq \frac{\pi}{2}$ and $\beta \simeq \frac{\pi}{8}$
- Hierarchy not explained, (but standard model-building methods can achieve that, e.g. F-N Mechanism)
- M_u and M_d exploit 7 pars to fit 10 observables with $\chi^2/\text{d.o.f} \simeq 1/2$
- Precise prediction of quark mass double ratio:

$$\frac{m_c}{m_t} \frac{m_b}{m_s} = \left| \frac{\lambda_u}{\lambda_d} \right|^2 = \tan^2 \frac{\pi}{8} (1 + \mathcal{O}(\lambda_0^2)) = 0.176 \pm 0.001$$

c.f. 0.177 ± 0.002 (exp)

- Precise predictions of UT angles:

- $\alpha - \frac{\pi}{2} = (1.30 \pm 0.02)^\circ$ c.f. $(1.6 \pm 1.4)^\circ$ (exp)
- $\beta - \frac{\pi}{8} = (-0.2 \pm 0.1)^\circ$ c.f. $(0.1 \pm 0.4)^\circ$ (exp)
- $\gamma - \frac{3\pi}{8} = (-1.1 \pm 0.1)^\circ$ c.f. $(-1.8 \pm 1.3)^\circ$ (exp)

Backup Slides

- Can re-write texture:

$$M_q^{HS} \equiv n_q \begin{pmatrix} c' \lambda_q^4 & b' \lambda_q^3 & 0 \\ b' \lambda_q^{*3} & b' \lambda_q^2 & A_0 \lambda_q^2 \\ 0 & A_0 \lambda_q^{*2} & 1 \end{pmatrix} \pm d \lambda_q^4 I$$

($b' \simeq b$). Still get good fit to data.

- First (leading) matrix solely responsible for quark mass differences and mixing parameters.
- Second (small) matrix is I_z -dependent “pedestal” on quark masses. Symmetric under a generation- $SU(3)$ symmetry.
- All coefficients (λ_0 , A_0 , b' , c' , d) symmetric under isospin reflection operator $u \leftrightarrow d$.
- Symmetry broken (only) by λ_q , n_q and the sign of d .

- We give here the algebraic NLO solutions of the texture:

$$\lambda = \lambda_0 (1 + f_\lambda \lambda_0^2) + \mathcal{O}(\lambda_0^5)$$

$$A = A_0 \left\{ 1 + \left[\frac{1}{4}(3b - 2\rho_0) - 2f_\lambda \right] \lambda_0^2 \right\} + \mathcal{O}(\lambda_0^4)$$

$$\bar{\rho} = \rho_0 (1 + c_0 f_\rho \lambda_0^2) + \mathcal{O}(\lambda_0^4)$$

$$\bar{\eta} = \eta_0 \left\{ 1 + \left[s_0 f_\rho + \frac{1}{2}(1 - 5b) \right] \lambda_0^2 \right\} + \mathcal{O}(\lambda_0^4),$$

where $f_\lambda = \frac{3}{4}f_A - \frac{5}{4} + \eta_0 \delta_c$,

$$f_A = \frac{1}{b} \left[A_0^2 + \frac{1}{2}(c_d + c_u) \right], \quad \delta_c = \frac{1}{b}(c_d - c_u)$$

and $f_\rho = \frac{1}{s_0} \left[-\frac{1}{2}f_A + \frac{7}{4}b - \frac{1}{2}\delta_c \right] + s_0(1 + \delta_c)$.

- NLO corrections above, as fractions of LO terms are respectively: -5.8×10^{-3} , $+2.6\%$, $+3.6\%$ and -1.8% (using fitted param values from table).

- For the quark mass ratios, we find:

$$\frac{m_1^q}{m_2^q} = -\lambda_q^2(1 - r_q) \left\{ 1 + \left[r_A \frac{(2-r_q)}{(1-r_q)} - 2 \right] \lambda_q^2 \right\} + \mathcal{O}(\lambda_q^6)$$

$$\frac{m_2^q}{m_3^q} = b\lambda_q^2 [1 + (1 - r_A)\lambda_q^2] + \mathcal{O}(\lambda_q^6),$$

where $r_q = \frac{c_q}{b}$ and $r_A = \frac{A_0^2}{b}$.

- NLO corrections to mass ratios m_c/m_t , m_s/m_b , m_u/m_c , m_d/m_s as fractions of LO terms are (resp.) -4.3×10^{-3} , -2.5% , $+4.3\%$, and $+4.5\%$ (using fitted param values from table).
- All results compatible with full numerical results reported in table.