Radiative processes at NLOPS accuracy

4th Liverpool Workshop on Muon Precision Physics 2025 - (MPP2025)

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On behalf of the BabaYaga team

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INFN National Institute for Nuclear Physics
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$e^+e^- \to \pi^+\pi^-$ channel

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$$a_\mu^{\text{HVP-LO}} = \left(\frac{\alpha\, m_\mu}{3\pi}\right)^2 \int_{m^2_{\pi_0}}^{\infty} ds\, \frac{1}{s^2} \mathcal{R}_0^{\text{had}}(s) \hat{\mathcal{K}}(s)$$

https://arxiv.org/pdf/2505.21476

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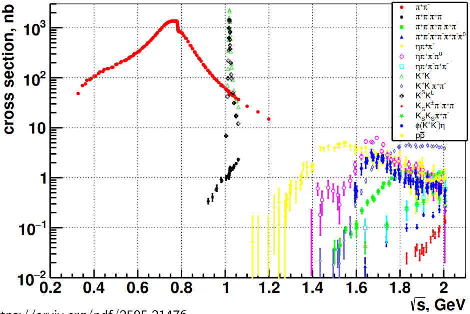

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$e^+e^- \to \pi^+\pi^-$

73% of $a_\mu^{\text{HVP-LO}}$

71% of $\delta(a_\mu^{\text{HVP-LO}})$

The Pion Form Factor

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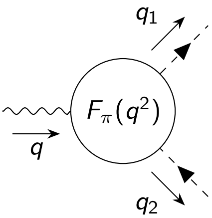
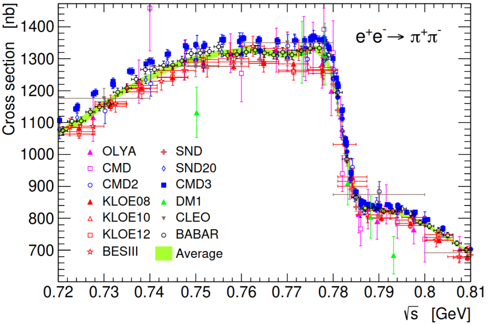
$$\langle \pi^+(q_2)\pi^-(q_1)|J^\mu_\pi(0)|0\rangle = e(q_1 - q_2)^\mu F_\pi\,(q^2)$$





https://arxiv.org/pdf/2312.02053

The energy scan

$$\sigma_0(e^+e^- \to \pi^+\pi^-, Q^2) \iff F_\pi\left(Q^2\right)$$

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$$\left|F_\pi\left(Q^2\right)\right|^2 \propto \frac{N_{\pi\pi}}{N_{ee}} \qquad R_{\mathrm{QED}} = \frac{N_{\mu\mu}}{N_{ee}}$$

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	$\mu\mu, ee$	$\pi\pi$
BabaYaga	NLOPS	NLOPS (GVMD, FsQED)
KKMC/BHWIDE	CEEX	CEEX
MCGPJ	NLO + CS	NLO + CS (GVMD)
McMule	NNLO	NLO (FsQED) + NNLO − ISC
SHERPA	NLO + YFS	NLO + YFS (sQED)

The radiative return

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$$\sigma(e^+e^- \to \pi^+\pi^-, s') \iff \sigma(e^+e^- \to \pi^+\pi^-\gamma)$$

Cfr. A. Denig's talk.

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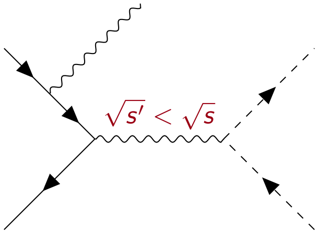

Codes currently available:

- McMule ($\mu\mu\gamma$ @NLO and $\pi\pi\gamma$ @NLO-ISC)
- Phokhara ($\mu\mu\gamma$, $\pi\pi\gamma$ @NLO)
- AfkQED ($\mu\mu\gamma$, $\pi\pi\gamma$ @LO + Collinear SF)

Cfr. A. Denig's talk.

Radiative processes

- An important ingredient: NLOPS approach
  - Exact calculation up to NLO.
  - Exclusive LL resummation to all orders with explicit photon kynematics.

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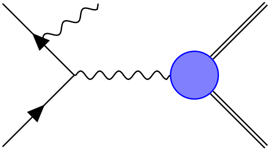
- Non perturbative contributions



$\times \quad \{F_\pi(M^2_{\pi\pi}), \alpha(M^2_{\pi\pi})\}$

LL approximation

$$\mathrm{d}\sigma_{\mathrm{LOPS}} = \sum_{\mathrm{n}=1}^{\infty} \Pi(\varepsilon, \mathrm{Q}^2) \frac{1}{\mathrm{n}!} \left|\mathcal{M}_{\mathrm{n}}^{\mathrm{X}}\right|^2 \mathrm{d}\Phi_{\mathrm{n}}(\{\mathrm{p}\}, \{\mathrm{k}\})$$

where

- $\Pi(\varepsilon, Q^2)$ is the Sudakov Form factor where ε is the soft-hard separator.
- $\mathcal{M}_n^X$ stands for $\mathcal{M}^{\mathrm{ex}}$ for $n = 1, 2$ and $\mathcal{M}^{\mathrm{LL}}$ for $n \geq 3$.

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$$\left|\mathcal{M}_n^{\mathrm{LL}}\right|^2 = \prod_{i=1}^{n-2}\left[-\frac{\alpha}{2\pi}P(z_i)\,\mathcal{I}_{\mathrm{SF}}(k_i, z_i)\,\frac{4\pi^2\,(1-z_i)}{z_i\omega_i^2}\right]\underbrace{\left|\mathcal{M}_2^{\mathrm{ex}}(\{\tilde{p}\},\{\tilde{k}\})\right|^2}_{e^+e^-\to XX\gamma\gamma}$$

LL approximation

If we define

$$\mathcal{I}(k) = \sum_{i,j=1}^{4} \mathcal{I}_{ij}(k) = \sum_{i,j=1}^{4} \eta_i \eta_j \frac{(p_i \cdot p_j)}{(p_i \cdot k)\,(p_j \cdot k)} k_0^2\,,$$

then

$$\mathcal{I}_{\rm SF}(k,z) = -\sum_{ij} \mathcal{I}_{ij}(k) \cdot F_{ij}(k,z)$$

where

$$\begin{cases} F_{ij}(k,z) = 1 - \delta_{ij} \dfrac{(1-z)^2}{1+z^2} & i = \text{fermion} \\ F_{ij}(k,z) = 1 & i = \text{scalar}\,. \end{cases}$$

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REMARK: In the limit $k \cdot p \to 0$ we correctly reproduce the massive unpolarized splitting function of *S. Dittmaier, arxiv:9904440*

Matching at NLO

$$\mathrm{d}\sigma^{\alpha}_{\mathrm{LL}} = \left[1 + \mathrm{C}_{\alpha,\mathrm{LL}}\right] \left|\mathcal{M}_1^{\mathrm{ex}}\right|^2 \mathrm{d}\Phi_1 + \left|\mathcal{M}_2^{\mathrm{ex}}\right|^2 \mathrm{d}\Phi_2$$

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- The hard contribution $|\mathcal{M}_2|^2$ is already exact.
- To match the exact $\mathrm{NLO\,soft-virtual\,correction}$ we introduce

$$F_{\mathrm{SV}} = 1 + (\mathcal{C}_{\alpha} - \mathcal{C}_{\alpha,\mathrm{LL}})$$

The master formula for NLOPS accuracy reads

$$\mathrm{d}\sigma_{\mathrm{NLOPS}} = \sum_{\mathrm{n}=1}^{\infty} \Pi(\varepsilon, \mathrm{Q}^2)\,\mathrm{F}_{\mathrm{SV}}\,\frac{1}{\mathrm{n}!}\,|\mathcal{M}_{\mathrm{n}}^{\mathrm{X}}|^2\,\mathrm{d}\Phi_{\mathrm{n}}(\{\mathrm{p}\},\{\mathrm{k}\})\,.$$

NNLO comparison with analytical results

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Considering the inclusive case over photon angles, *V.S. Fadin and R.N. Lee (arxiv:2308.09479)* provides the full expression for the NNLO SS+SV+VV correction for the process $e^+e^- \to \gamma\gamma^*$(ISC).

$$\mathrm{d}\sigma_{\mathrm{ss+sv+vv}} = \mathrm{c}_0 + \mathrm{c}_1\mathrm{L} + \mathrm{c}_2\mathrm{L}^2 + \mathrm{c}_3\mathrm{L}^3 + \mathrm{c}_4\mathrm{L}_\omega + \mathrm{c}_5\mathrm{L}\mathrm{L}_\omega + \mathrm{c}_6\mathrm{L}^2\mathrm{L}_\omega + \mathrm{c}_7\mathrm{L}_\omega^2 + \mathrm{c}_8\mathrm{L}\mathrm{L}_\omega^2 + \mathrm{c}_9\mathrm{L}^2\mathrm{L}_\omega^2$$

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- A comparison with the $\mathcal{O}(\alpha^2)$ expansion (w.r.t. the radiative process) of the master formula for the 1γ sample shows the following.

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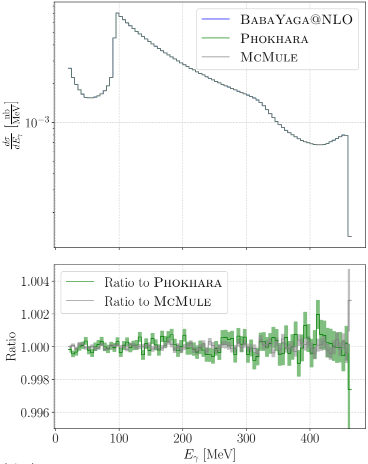
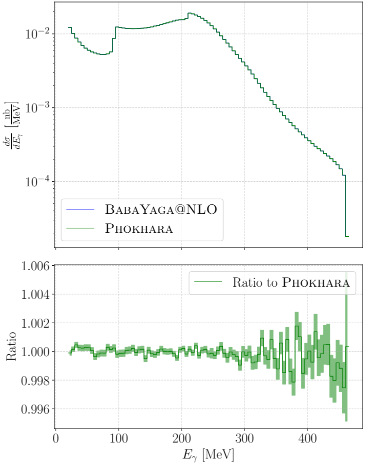
- A comparison with the $\mathcal{O}(\alpha^2)$ expansion (w.r.t. the radiative process) of the master formula for the 1γ sample shows the following.

- Virtual corrections on the 2γ sample are correctly reproduced in the soft limit, due to YFS theorem.

Tuned comparison - E_γ

Scenario KLOE-LA @ 1.02 GeV





arxiv.org/abs/2410.22882

Matrix element comparison for $n \geq 3$

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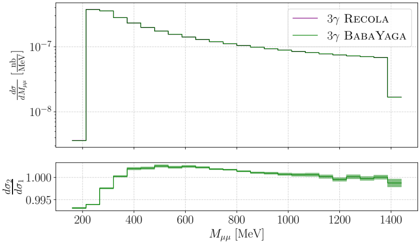
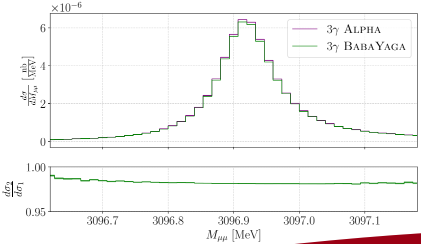
Scenario B with 2 additional photons with $E_\gamma \geq 100$ MeV (Tree-level)





ALPHA arXiv:hep- ph/9507237
RECOLA arxiv.org/abs/1211.6316

ε independence

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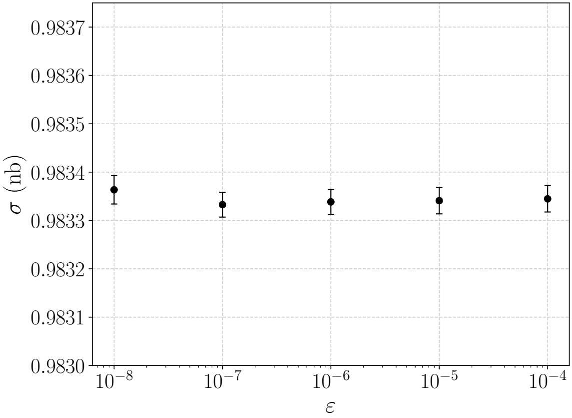

KLOE-LA scenario - $M_{\mu\mu,\pi\pi}$

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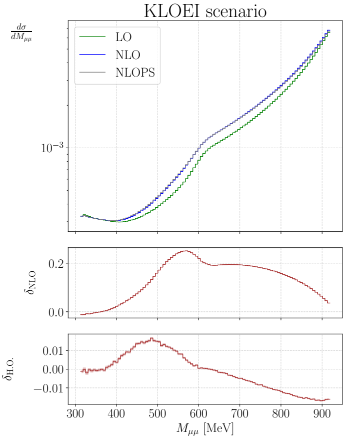
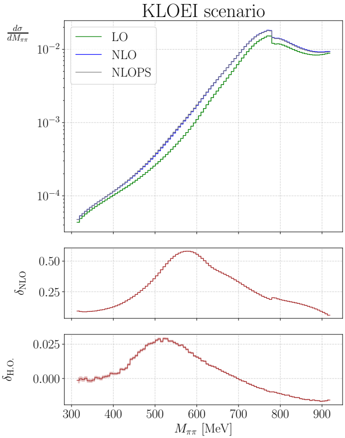
$$\delta_{\mathrm{NLO}} = \frac{d\sigma_{\mathrm{NLO}} - d\sigma_{\mathrm{LO}}}{d\sigma_{\mathrm{LO}}} \qquad \delta_{\mathrm{H.O.}} = \frac{d\sigma_{\mathrm{NLOPS}} - d\sigma_{\mathrm{NLO}}}{d\sigma_{\mathrm{NLO}}}$$





KLOE-SA scenario - $M_{\mu\mu,\pi\pi}$

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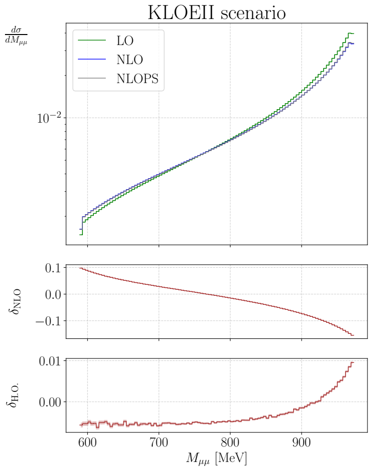
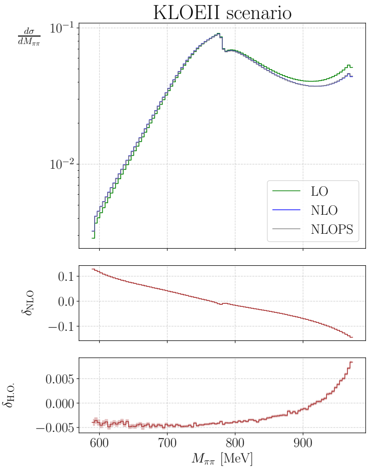
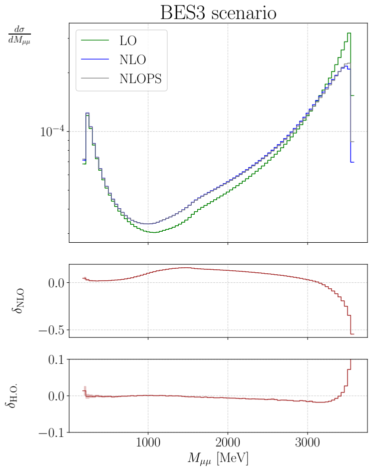
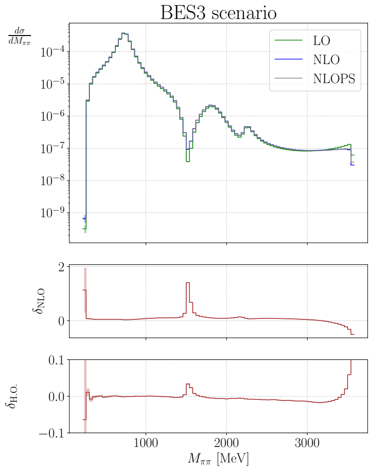


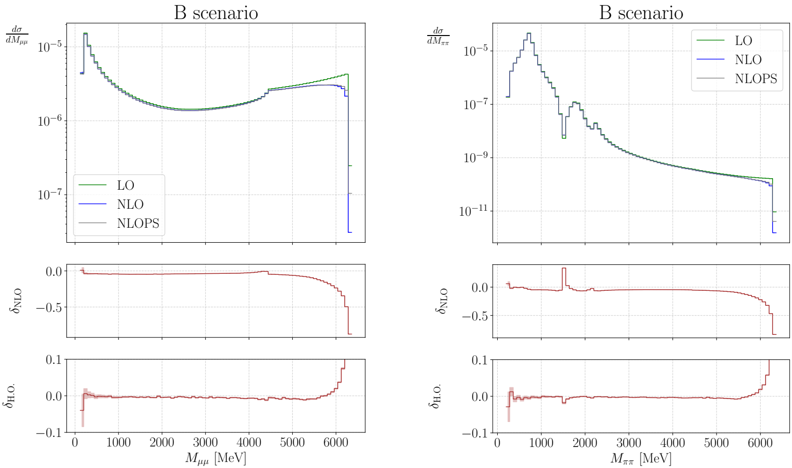


BES3 scenario - $M_{\mu\mu,\pi\pi}$



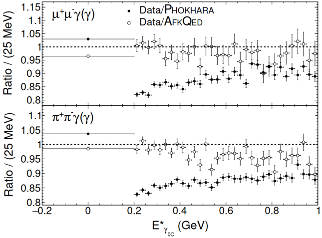


B scenario - $M_{\mu\mu,\pi\pi}$



BaBar test (arXiv:2308.05233)

- $E^*_\gamma > 4\,\mathrm{GeV}$ and $0.35 < \theta_\gamma < 2.4$ rad
- $p_\perp > 0.1\,\mathrm{GeV}$ and $0.4 < \theta_\pm < 2.45$ rad
- $M_{\mu\mu} < 1.4\,\mathrm{GeV}$ and $0.6 < M_{\pi\pi} < 0.9$ GeV
- p_{0c} missing momentum with $E^*_{0c} > 200$ MeV



Cfr. A. Denig's talk.

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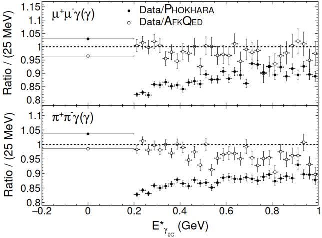
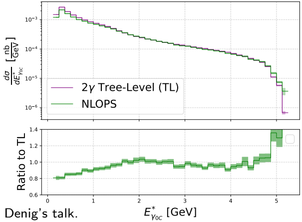


$\pi\pi\gamma$



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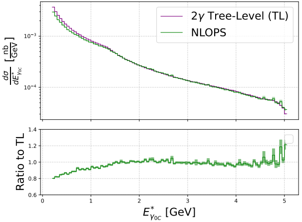

Cfr. A. Denig's talk.

Summary and Outlook

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  $$\mathrm{NLOPS} \quad \mathrm{vs} \quad \mathrm{NNLO} \Rightarrow \text{Assessment of theoretical uncertainty}$$

- A preliminary version of BabaYaga@NLO for $\mu\mu\gamma$ and $\pi\pi\gamma$ is working (with room for improvement in the near future).

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- Multiple photon contributions are in general important for radiative return experiment.

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  NLOPS vs NNLO $\Rightarrow$ Assessment of theoretical uncertainty

- A preliminary version of BabaYaga@NLO for $\mu\mu\gamma$ and $\pi\pi\gamma$ is working (with room for improvement in the near future).

- Future developments
  - $\pi\pi\gamma$ with FsQED and GVMD at NLOPS accuracy.
  - $ee\gamma$ at NLOPS accuracy.
  - Extension to other hadronic channels ($K^+K^- \dots$)

Thank you !

Scenarios Strong2020

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(a) KLOEI Large-Angle scenario

$$\sqrt{s} = 1.02\,\mathrm{GeV}$$
$$50^\circ \le \vartheta_\pm \le 130^\circ$$
$$|\mathbf{p}^z_\pm| \ge 90\,\mathrm{MeV} \ \vee \ |\mathbf{p}^\perp_\pm| \ge 160\,\mathrm{MeV}$$
$$50^\circ \le \vartheta_\gamma \le 130^\circ \ \wedge \ E_\gamma \ge 20\,\mathrm{MeV}$$
$$0.1\,\mathrm{GeV}^2 \le M^2_{XX} \le 0.85\,\mathrm{GeV}^2$$

(b) KLOEII Small-Angle scenario

$$\sqrt{s} = 1.02\,\mathrm{GeV}$$
$$50^\circ \le \vartheta_\pm \le 130^\circ$$
$$|\mathbf{p}^z_\pm| \ge 90\,\mathrm{MeV} \ \vee \ |\mathbf{p}^\perp_\pm| \ge 160\,\mathrm{MeV}$$
$$\vartheta_{\tilde{\gamma}} \le 15^\circ \ \vee \ \vartheta_{\tilde{\gamma}} \ge 165^\circ$$
$$0.35\,\mathrm{GeV}^2 \le M^2_{XX} \le 0.95\,\mathrm{GeV}^2$$

(c) BES3 scenario

$$\sqrt{s} = 4\,\mathrm{GeV}$$
$$|\cos\vartheta_\pm| \le 0.93 \ \wedge \ |p^\perp_\pm| \ge 300\,\mathrm{MeV}$$
$$[|\cos\vartheta_\gamma| \le 0.8 \ \wedge \ E_\gamma \ge 25\,\mathrm{MeV}] \ \vee$$
$$[0.86 \le |\cos\vartheta_\gamma| \le 0.92 \ \wedge \ E_\gamma \ge 50\,\mathrm{MeV}]$$
$$\exists!\ \gamma \ \text{ s.t. } \ E_\gamma \ge 400\,\mathrm{MeV}$$

(d) B scenario

$$\sqrt{s} = 10\,\mathrm{GeV}$$
$$0.65\,\mathrm{rad} \le \vartheta_\pm \le 2.75\,\mathrm{rad} \ \wedge \ |\mathbf{p}_\pm| \ge 1\,\mathrm{GeV}$$
$$0.6\,\mathrm{rad} \le \vartheta_\gamma \le 2.7\,\mathrm{rad} \ \wedge \ \mathrm{E}_\gamma \ge 3\,\mathrm{GeV}$$
$$\vartheta_{\tilde{\gamma},\gamma^{(h)}} \le 0.3\,\mathrm{rad}$$
$$M_{XX\gamma} \ge 8\,\mathrm{GeV}$$

Tuned comparison - KLOE-SA

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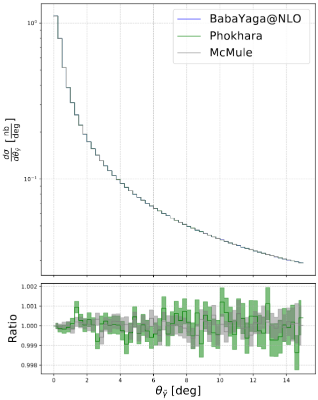
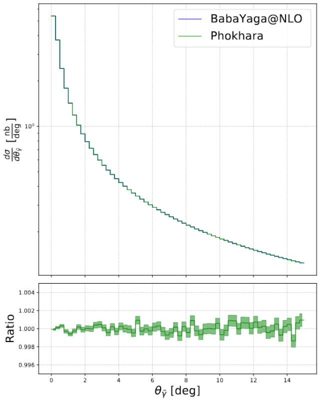
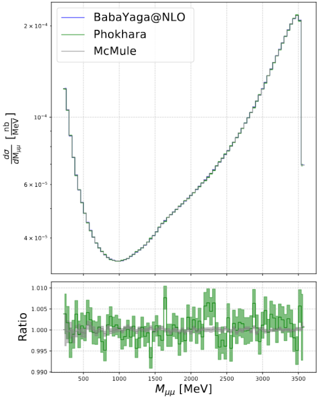
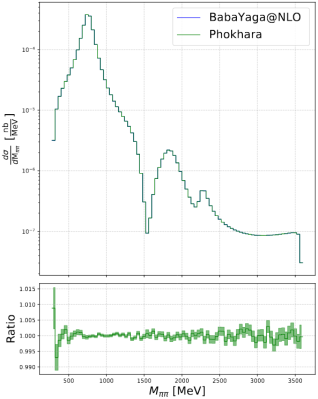


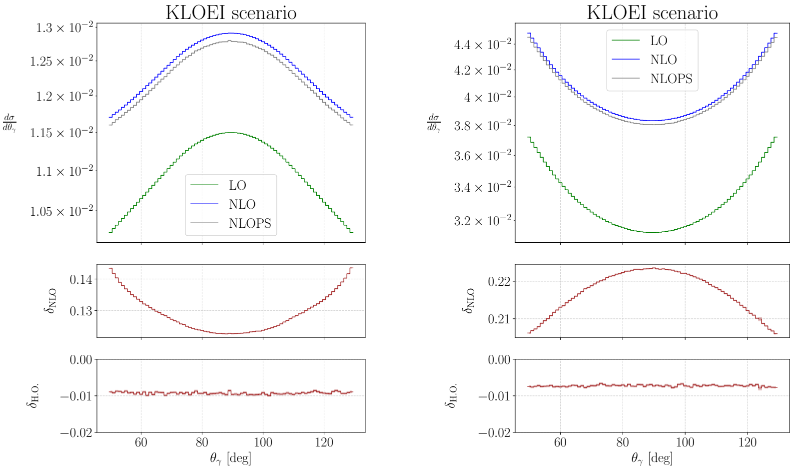


Tuned comparison - BES3

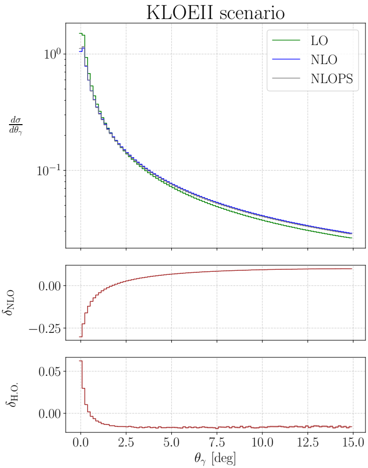


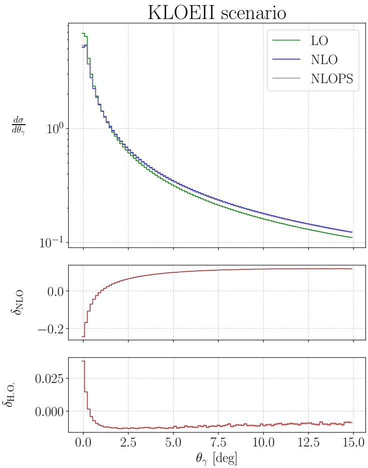


KLOE-LA scenario - θ_γ



KLOE-SA scenario - $\theta_{\tilde{\gamma}}$





KLOE-LA scenario - E_γ

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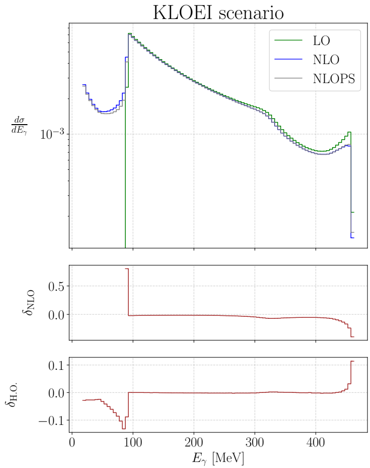
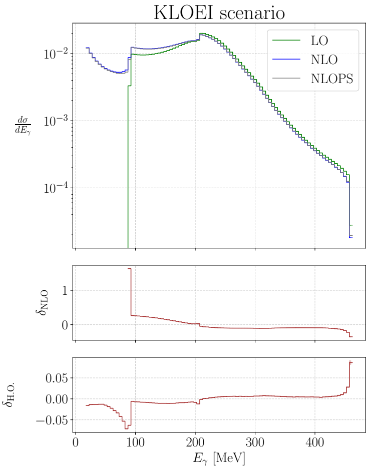
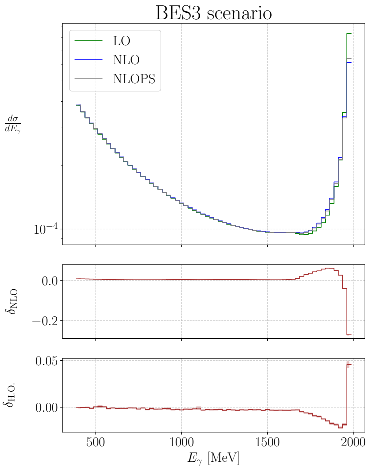
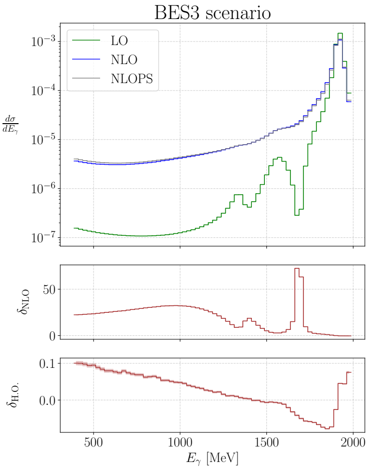




BES3 scenario - E_γ





B scenario - E_γ