Improved calculation of radiative corrections to $\tau \to \pi\pi\nu_\tau$ decays

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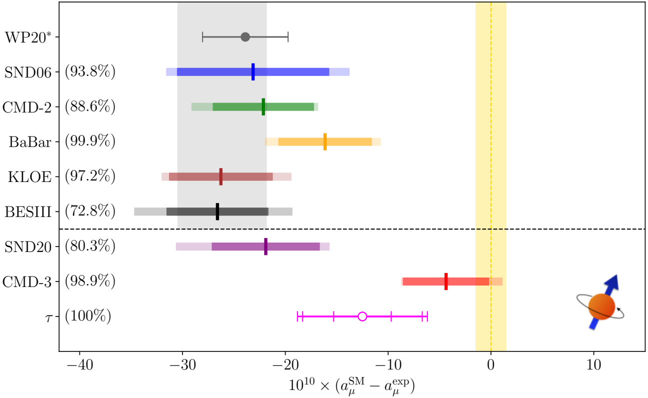
Colangelo, Cottini, MH, Holz, arXiv:2510.26871, 2511.07507

Data-driven determinations of HVP



- Confusing situation in $e^+e^- \to \pi^+\pi^-$ Talks by T. Teubner, A. Wright, A. Denig
- Can we get $\tau \to \pi\pi\nu_\tau$ theory under control to justify **τ-based HVP evaluations**?

Master formula for $\tau \to \pi\pi\nu_\tau(\gamma)$

$$\frac{1}{K_\Gamma(s)}\frac{d\Gamma}{ds}[\tau \to \pi\pi\nu_\tau(\gamma)] = \underbrace{S_{\text{EW}}^{\pi\pi}}_{\text{short distance}} \times \underbrace{\left[\beta_{\pi\pi^0}\right]^3}_{\text{phase space}} \times \underbrace{|f_+(s)|^2}_{\langle\pi\pi^0|j_W^\mu|0\rangle} \times \underbrace{G_{\text{EM}}(s)}_{\text{radiative corrections}}$$

- Alternative approach via **hadronic τ decays**: $\tau \to h\nu_\tau$, $h = 2\pi, 4\pi, \ldots$ related to $I = 1$ part of $e^+e^- \to h$ cross section Alemany et al. 1998
- **Experimental status**: LEP and Belle, new data from Belle II
- Relation exact in limit of **isospin symmetry**

  $\hookrightarrow$ need to control corrections, especially in $\underbrace{\langle\pi^+\pi^-|j_{\text{em}}^\mu|0\rangle}_{F_\pi^V(s)}$ vs. $\underbrace{\langle\pi^\pm\pi^0|j_{W\mp}^\mu|0\rangle}_{f_+(s)}$
- **Isospin breaking (IB)**: corrections to CVC important, especially $f_+(s)$ vs. $F_\pi^V(s)$

Isospin-breaking corrections to $\tau \to \pi\pi\nu_\tau$: basics

Master formula for $\tau \to \pi\pi\nu_\tau(\gamma)$

$$\frac{1}{K_\Gamma(s)}\frac{d\Gamma}{ds}[\tau \to \pi\pi\nu_\tau(\gamma)] = \underbrace{S_{\text{EW}}^{\pi\pi}}_{\text{short distance}} \times \underbrace{\beta^3_{\pi\pi^0}}_{\text{phase space}} \times \underbrace{|f_+(s)|^2}_{\langle\pi\pi^0|j_W^\mu|0\rangle} \times \underbrace{G_{\text{EM}}(s)}_{\text{radiative corrections}}$$

- Short-distance corrections

$$S_{\text{EW}}^{\pi\pi} = 1 + \frac{2\alpha}{\pi}\log\frac{M_Z}{m_\tau} + \cdots$$

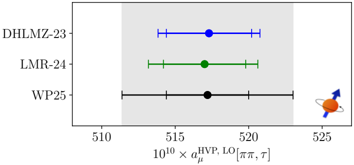
- In isospin limit: $f_+(s)$ same as $F_\pi^V(s)$ (matrix element from $e^+e^- \to \pi^+\pi^-$)
- Radiative corrections subsumed into $G_{\text{EM}}(s)$ (similar to $\eta(s)$ in $e^+e^- \to \pi^+\pi^-$)

$$\sigma_{e^+e^-\to\pi^+\pi^-(\gamma)}(s) = \frac{1}{\mathcal{N}(s)\Gamma_e}\frac{d\Gamma_{\tau^\pm\to\pi^\pm\pi^0\nu_\tau(\gamma)}}{ds} \times \frac{1+\frac{\alpha}{\pi}\eta(s)}{S_{\text{EW}}^{\pi\pi}G_{\text{EM}}(s)}\frac{\big[\beta_{\pi\pi}(s)\big]^3}{\big[\beta_{\pi\pi^0}(s)\big]^3}\left|\frac{F_\pi^V(s)}{f_+(s)}\right|^2$$

- Procedure:
  1. Remove τ-specific IB corrections: $S_{\text{EW}}^{\pi\pi}G_{\text{EM}}(s)$ and phase space
  2. Apply corrections to matrix element to get from $f_+(s)$ to $F_\pi^V(s)$
  3. Add e^+e^- specific IB corrections ($\eta(s)$ and ρ–ω mixing)

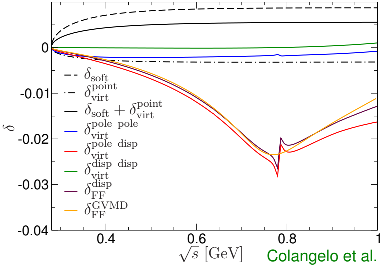
Isospin-breaking corrections to $\tau \to \pi\pi\nu_\tau$: status

		Refs. [168, 196]	Ref. [211]	Refs. [239, 249]	Our estimate
Phase space		-7.88	-7.52	–	$-7.7(2)$
$S_{\rm EW}$		$-12.21(15)$	$-12.16(15)$	–	$-12.2(1.3)$
$G_{\rm EM}$		$-1.92(90)$	$(-1.67)^{+0.60}_{-1.39}$	–	$-2.0(1.4)$
FSR		$4.67(47)$	$4.62(46)$	$4.42(4)$	$4.5(3)$
ρ–ω mixing		$4.0(4)$	$2.87(8)$	$3.79(19)$	$3.9(3)$
$\frac{F_\pi^V}{f_+}$ (w/o ρ–ω)	ΔM_ρ	$0.20(^{+27}_{-19})(9)$	$1.95^{+1.56}_{-1.55}$	–	
	$\Delta\Gamma_\rho(\Delta M_\pi)$	$4.09(0)(7)$	3.37	–	
	$\Delta\Gamma_\rho(\pi\pi\gamma)$	$-5.91(59)(48)$	$-6.66(73)$	–	
	$\Delta\Gamma_\rho(g_{\rho\pi\pi})$	–	–	–	
	Total	$-1.62(65)(63)$	$(-1.34)^{+1.72}_{-1.71}$	–	$-1.5(4.7)$
Sum		$-14.9(1.9)$	$(-15.20)^{+2.26}_{-2.63}$	–	$-15.0(5.1)$

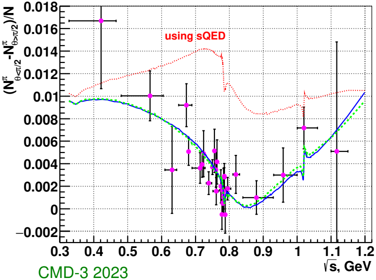


- Status of IB corrections
  - $\text{FSR}(s) = 1 + \frac{\alpha}{\pi}\eta(s)$ and ρ–ω mixing from $e^+e^- \to \pi^+\pi^-$
    $\hookrightarrow$ reasonably well under control
  - Otherwise, uncertainty currently difficult to quantify, attempt made in WP25
- Main challenges:
  1. **Short-distance matching**: $\mathcal{O}(\frac{\alpha}{\pi})$ uncertainty beyond LL Work in progress, see below
  2. **Long-range radiative corrections**: structure-dependent effects in $G_{\text{EM}}(s)$ This talk
  3. **IB in matrix elements**: $f_+(s)$ vs. $F_\pi^V(s)$ Work in progress, Colangelo, Cottini, Ruiz de Elvira 2025

Why worry about structure-dependent radiative corrections?



Colangelo et al. 2022



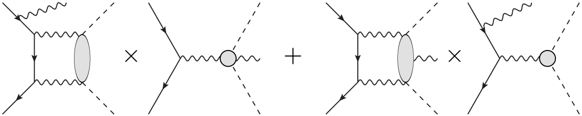
CMD-3 2023

- CMD-3 found large deviations from MC result for **forward–backward asymmetry**

  $\hookrightarrow$ understood from **resonance enhancement of virtual corrections** Ignatov, Lee 2022

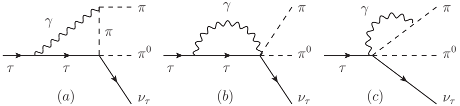
- Potentially relevant for ISR experiments Ignatov 2021

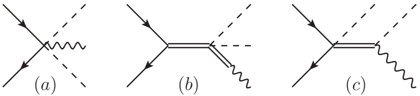
  $\hookrightarrow$ **ISR–FSR interference**:



- Under investigation RadioMonteCarLow 2, see Friday

Calculation in ChPT





- Applies at low energies
- UV divergences removed by LECs

$$X_\ell \equiv \frac{4}{3}X_1 + X_6^r(\mu_\chi) - 4K_{12}^r(\mu_\chi)$$

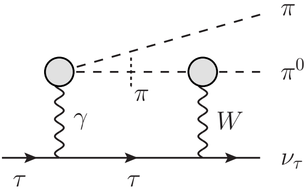
$\hookrightarrow$ finite parts not predicted

- **Matching to lattice-QCD calculations** of $\langle \pi\pi^0 | j^\mu_{\text{em}} j^\nu_W | 0\rangle$ Feng et al. 2020, Yoo et al. 2023
- For now use $X_\ell(M_\rho) = 14 \times 10^{-3}$ Ma et al. 2021

$\hookrightarrow$ improved matching in preparation Cirigliano et al.

- Local ChPT contribution dropped in previous work (including $\Delta X_\ell\big|_{\text{SD}} = -\frac{1}{4\pi^2}\log\frac{m_\tau^2}{M_\rho^2}$)
- Uncertainty in real emission dominated by F_A

Dispersive calculation



- Dominant correction from **pion pole** in $\langle \pi\pi^0 | j^\mu_{\text{em}} j^\nu_W | 0 \rangle$

  $\hookrightarrow$ reduces to ChPT for point-like form factors

- **Matching at low energies**

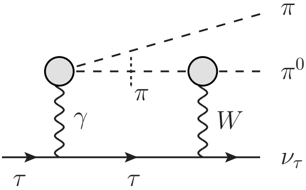
$$f^{\text{full}}_{\text{loop}}(s,t) = f^{\text{disp}}_{\text{loop}}(s,t) - f^{\text{disp}}_{\text{loop}}(0,0) + f^{\text{ChPT}}_{\text{loop}}(0,0)$$

  $\hookrightarrow$ ensures that IR structure and chiral logs are correct

- Checked that limits match onto ChPT, including narrow-width limit and $M_\rho \to \infty$ for UV divergence

- This is the **same topology** as in the $e^+e^- \to \pi^+\pi^-$ asymmetry!

Some technicalities of the box diagram



- Use an unsubtracted dispersion relation

$$f_+(s) = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \, \frac{\operatorname{Im} f_+(s')}{s' - s}$$

$\hookrightarrow$ result UV finite

- Interpret Cauchy kernel as loop propagator
- Express the integral in terms of standard Passarino–Veltman functions

$$f_{\text{loop}}^{\text{disp}}(s,t) = \alpha \int_{4M_\pi^2}^{\infty} ds' \int_{4M_\pi^2}^{\infty} ds'' \, \operatorname{Im} f_+(s') \operatorname{Im} f_+(s'') \sum_{k \in \{B_0, C_0, D_0\}} \mathcal{M}_k(s,t,s',s'')$$

- IR and endpoint singularities need to be treated carefully

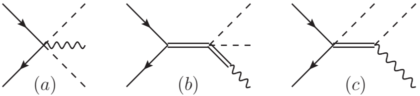
Real emission: bremsstrahlung off τ and π

- **Leading Low term**
  - Cancels IR divergence
  - Logarithmic divergence at threshold
- **Remaining radiation off τ and π**
  - Exhibits threshold divergence $G_{\text{EM}}(s) \propto 1/(s-4M_\pi^2)$
  - Numerically largest effect
  - Delicate to evaluate close to threshold
  - Developed parameterization via suitably chosen angles that allows for stable evaluation down to threshold
- **Threshold enhancement** makes certain $\mathcal{O}(e^4)$ effects relevant, otherwise, always work at $\mathcal{O}(e^2)$, ensure correct threshold by mapping

$$G_{\text{EM}}(s) \to G_{\text{EM}}\big[\tilde{s}(s)\big]$$

$$\tilde{s}(s) = \frac{(m_\tau^2 - 4M_\pi^2)s + \big[4M_\pi^2 - (M_\pi + M_{\pi^0})^2\big]m_\tau^2}{m_\tau^2 - (M_\pi + M_{\pi^0})^2} \qquad \tilde{s}\big[(M_\pi + M_{\pi^0})^2\big] = 4M_\pi^2 \qquad \tilde{s}(m_\tau^2) = m_\tau^2$$

Real emission: resonance diagrams



- Keep states required for **resonance saturation** of L_9, L_{10} + WZW anomaly

  $\hookrightarrow$ free parameters F_V, G_V, F_A

- **Short-distance constraints**

$$F_V = \sqrt{2}F_\pi \simeq 0.13\,\text{GeV} \qquad G_V = \frac{F_\pi}{\sqrt{2}} \simeq 0.065\,\text{GeV} \qquad F_A = F_\pi \simeq 0.092\,\text{GeV}$$

- **Phenomenological determinations** from $\rho \to e^+e^-, \pi\pi$, $K^* \to K\pi$, $a_1 \to \pi\gamma$

$$F_V \simeq 0.16\,\text{GeV} \qquad G_V \simeq 0.065\,\text{GeV} \qquad F_A \simeq 0.12\,\text{GeV}$$

- We tried a new estimate $a_1 \to \pi\rho \to \pi\gamma$, yielding $F_A = (0.07\ldots0.13)\,\text{GeV}$

- Uncertainty in F_A dominant effect, little motivation to include higher multiplets

Dispersive representation of pion form factor

- Need input for $\mathrm{Im}\, f_+(s)$, should be consistent with $\tau \to \pi\pi\nu_\tau$ spectrum
- Follow strategy from Colangelo, MH, Stoffer 2018 …, supplemented by ρ', ρ''

$$f_+(s) = \Big[1 + G_{\text{in}}^N(s) + \sum_{V=\rho',\rho''} c_V \mathcal{A}_V(s)\Big]\Omega_1^1(s) \qquad \Omega_1^1(s) \equiv \exp\bigg\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)}\bigg\}$$

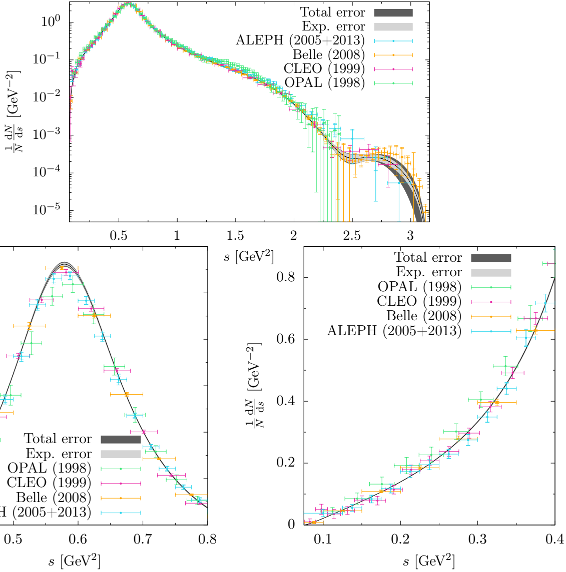
  - P-wave phase shift $\delta_1^1(s)$ from Roy equations, free parameters: $\delta_1^1(s_0)$, $\delta_1^1(s_1)$
  - Conformal polynomial with $\pi\omega$ threshold, constrained as P wave and by $f_+(0) = 1$
  - Resonance terms

$$\mathcal{A}_V(s) = \frac{s}{\pi}\int_{s_{\text{thr}}}^\infty ds' \frac{\mathrm{Im}\,\mathcal{A}_V(s')}{s'(s'-s)} \qquad \mathrm{Im}\,\mathcal{A}_V(s) = \mathrm{Im}\,\frac{1}{M_V^2 - s - i\sqrt{s}\Gamma_V(s)}$$

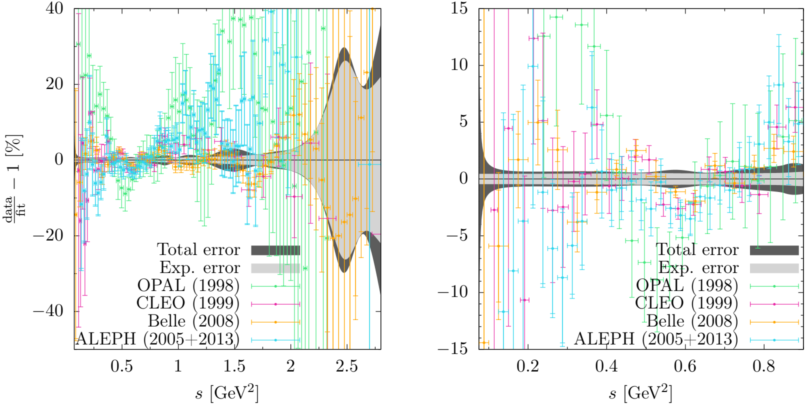
  - Total number of parameters: $2 + 3 \times 2 + N - 2 = 6 + N$

- Calculate first approximation for $G_{\text{EM}}(s)$ with $f_+(s) = \Omega_1^1(s)$, iterate until convergence (few steps)

Fits to the τ spectrum

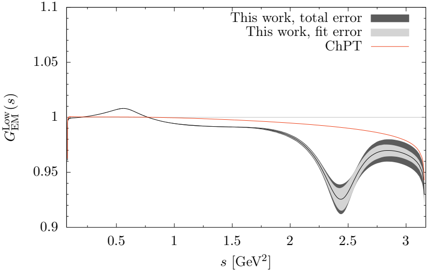


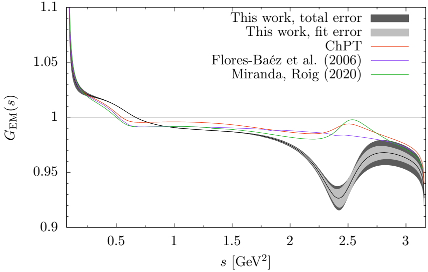
Fits to the τ spectrum



- Observations
  - Fits are not perfect, tensions among the data sets do exist
  - Tension between threshold and $\rho(770)$ region upon imposing analyticity/unitarity constraints
  - **New data from Belle II would be extremely valuable!**

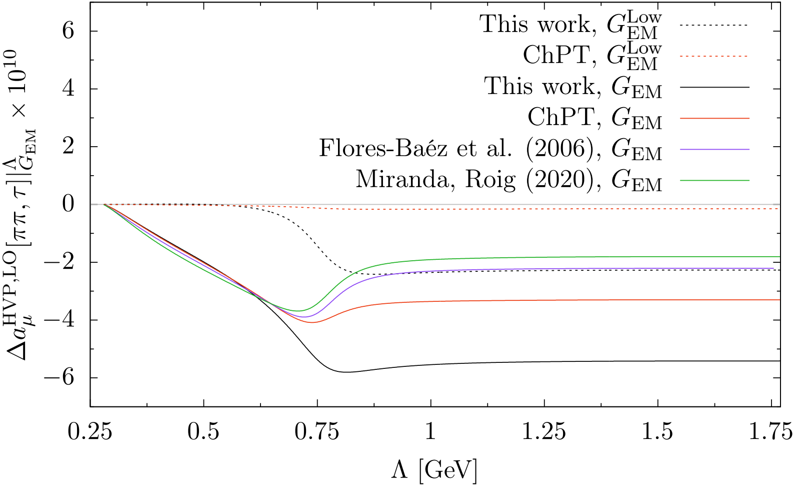
Results





	DHLMZ-23	LMR-24	WP25	This work
Phase space	-7.88	-7.52	$-7.7(2)$	$-7.74(5)$
$S_{\text{EW}}^{\pi\pi}$	$-12.21(15)$	$-12.16(15)$	$-12.2(1.3)$	$-12.2(1.3)$
$G_{\text{EM}}^{\text{full}}$	$-1.92(90)$	$(-1.67)^{+0.60}_{-1.39}$	$-2.0(1.4)$	$-5.4(5)$
Sum	$-22.01(91)$	$(-21.35)^{+0.62}_{-1.40}$	$-21.9(1.9)$	$-25.3(1.4)$
Full	–	–	–	$-24.8(1.4)$

Results



This work, $G_{\mathrm{EM}}^{\mathrm{Low}}$
ChPT, $G_{\mathrm{EM}}^{\mathrm{Low}}$
This work, G_{EM}
ChPT, G_{EM}
Flores-Baéz et al. (2006), G_{EM}
Miranda, Roig (2020), G_{EM}
$\Delta a_\mu^{\mathrm{HVP,LO}}[\pi\pi,\tau]|^{\Lambda}_{G_{\mathrm{EM}}} \times 10^{10}$
6
4
2
0
-2
-4
-6
0.25
0.5
0.75
1
1.25
1.5
1.75
Λ [GeV]

Towards an improved short-distance matching for $\tau \to \pi\pi\nu_\tau$

- Standard factor $S_{\text{EW}} = 1 + \frac{2\alpha}{\pi} \log \frac{M_Z}{m_\tau} + \cdots$ encodes **universal corrections** to all semi-leptonic decays, known at NLL (in a particular scheme)
- However: not accounted for at present is **scheme dependence** of the short-distance Wilson coefficient at NLL

  $\hookrightarrow$ corresponds to choice of evanescent operator Cirigliano et al. 2023
- Needs to cancel in the **matching at the hadronic scale**

  $\hookrightarrow$ organized at the level of the chiral LEC
- Matching can be performed, following Descotes-Genon 2005, Cirigliano et al. 2023, based on lattice-QCD input for γW box correction Feng et al. 2020, Yoo et al. 2023
- Further cross check: in addition to evanescent scheme, need to show that all scale dependence cancels at the order considered Work in progress
  - Chiral renormalization scale μ_χ
  - LEFT renormalization scale μ

A similar example: radiative corrections to neutron β decay

- Start from **LEFT Lagrangian**

$$\mathcal{L}_{\text{LEFT}} = -2\sqrt{2}G_F\bar{e}_L\gamma_\rho\mu_L\,\bar{\nu}_{\mu L}\gamma^\rho\nu_{eL} - 2\sqrt{2}G_F V_{ud}C(a,\mu)\bar{e}_L\gamma_\rho\nu_{eL}\,\bar{u}_L\gamma^\rho d_L + \text{h.c.} + \cdots$$

$\hookrightarrow$ scheme for G_F defined by muon decay

- Wilson coefficient for the semileptonic operator in $\overline{\text{MS}}$ + NDR for γ_5

$$C(a,\mu) = 1 + \frac{\alpha}{\pi}\log\frac{M_Z}{\mu} + \underbrace{\frac{\alpha}{\pi}\left(\frac{a}{6}-\frac{3}{4}\right)}_{\equiv B(a)} - \frac{\alpha\alpha_s}{4\pi^2}\log\frac{M_W}{\mu} + \mathcal{O}(\alpha\alpha_s,\alpha^2)$$

$$\gamma^\alpha\gamma^\rho\gamma^\beta P_L\otimes\gamma_\beta\gamma_\rho\gamma_\alpha P_L = 4\big[1+a(4-d)\big]\gamma^\rho P_L\otimes\gamma_\rho P_L + E(a)$$

- Dependence on a and μ needs to cancel in observables (at the order considered)

- **Matching to ChPT**: relevant low-energy constant is $g_V(\mu_\chi) = 1 + \mathcal{O}(\alpha)$

$\hookrightarrow$ corrections correspond to $X_\ell(\mu_\chi)$ here

A similar example: radiative corrections to neutron β decay

- Master formula Cirigliano et al. 2023

$$g_V(\mu_\chi) = \bar{C}(\mu)\left[1 + \bar{\square}^V_{\text{had}}(\mu_0) - \frac{\alpha(\mu_\chi)}{2\pi}\left(\frac{5}{8} + \frac{3}{4}\log\frac{\mu_\chi^2}{\mu_0^2} + \left(1 - \frac{\alpha_s(\mu_0)}{4\pi}\right)\log\frac{\mu_0^2}{\mu^2}\right)\right]$$

$$C(a,\mu) \equiv \bar{C}(\mu)\left(1 + \frac{\alpha(\mu)}{\pi}B(a)\right)$$

$$\bar{\square}^V_{\text{had}}(\mu_0) \equiv -ie^2\int\frac{d^4q}{(2\pi)^4}\frac{\nu^2+Q^2}{Q^4}\left[\frac{T_3(\nu,Q^2)}{2m_N\nu} - \frac{2}{3}\frac{1}{Q^2+\mu_0^2}\left(1 - \frac{\alpha_s(\mu_0)}{\pi}\right)\right]$$

- $T_3(\nu, Q^2)$ is a two-current matrix element of the nucleon, μ_0 another factorization scale
- Dependence on a, μ, μ_χ, μ_0 drops out from decay rate at the considered order
- For $\tau \to \pi\pi\nu_\tau$, the analog of $T_3(\nu, Q^2)$ can be extracted from existing lattice calculations Feng et al. 2020, Yoo et al. 2023

General strategy for the calculation of IB corrections

- Strategy for phenomenological calculations:
  1. Starting point: **chiral perturbation theory**

     $\hookrightarrow$ validity limited to low-energy region
  2. Combination with **dispersion relations**

     $\hookrightarrow$ include resonance effects, match to ChPT
  3. Precision often limited by low-energy constants (LECs)

     $\hookrightarrow$ combination with **lattice QCD**

- **Choice of isospin scheme often hidden in LECs** Gasser, Rusetsky, Scimemi 2003

- Examples:
  - IB in pion form factor $F_\pi^V(s)$ Monnard 2021, Colangelo et al. 2025, work in progress

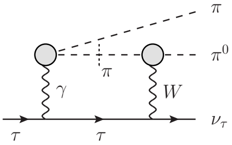
    $\hookrightarrow$ need to match to ChPT for subtraction constants
  - IB in $e^+e^- \to 3\pi$ MH, Hoid, Kubis, Schuh 2023, Biloshytskyi et al. 2025

    $\hookrightarrow$ need to choose an isospin-limit value of $\omega \to 3\pi$ coupling
  - Similar case: $\rho^\pm \to \pi^\pm\pi^0$ vs. $\rho^0 \to \pi^+\pi^-$ coupling for τ IB corrections WP25
  - **Radiative corrections to $\tau \to \pi\pi\nu_\tau$** Colangelo, Cottini, MH, Holz 2025

Conclusions and outlook

- Calculation of **IB corrections** crucial in multiple instances for $(g-2)_\mu$ program
  - Radiative corrections to $e^+e^- \to \pi^+\pi^-(\gamma)$
  - Hadronic τ decays
  - Detailed comparisons of HVP calculations to lattice QCD
- From phenomenological perspective: **ChPT + dispersion relations**

  $\hookrightarrow$ often limited by LECs, or implicit choice of IB scheme
- Obvious case for **complementarity with lattice QCD**
- **Radiative corrections to $\tau \to \pi\pi\nu_\tau$**
  - Extended validity of $G_{\text{EM}}(s)$ beyond low-energy region
  - Structure-dependent virtual corrections are large (again)
  - Improved matching to ChPT and short-distance corrections In progress



- Main open point for τ decays: $f_+(s)$ vs. $F_\pi^V(s)$
  - Can be addressed with dispersion relations
  - Expect important role of subtraction constants
  - **Matching to ChPT and lattice QCD to be developed!**