Rescattering corrections to the pion Compton tensor

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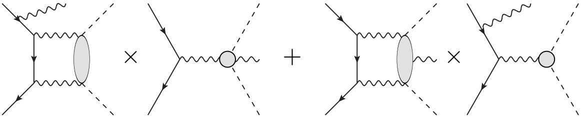
based on MH, Stoffer arXiv:1905.13198, Colangelo, MH, Stoffer, Procura arXiv:1309.6877, arXiv:1506.01386

Motivation



- **Structure-dependent radiative corrections** potentially sizable in $e^+e^- \to \pi^+\pi^-\gamma(\gamma)$
- Two recent examples
  - Forward–backward asymmetry in $e^+e^- \to \pi^+\pi^-$
  - Radiative corrections to $\tau \to \pi\pi\nu_\tau$
- Required matrix elements
  1. $\gamma^*\gamma^* \to \pi\pi$: construction of matrix elements known, but how to combine with multiloop techniques and implement in MC generators? This talk
  2. $\gamma^*\gamma^*\gamma \to \pi\pi$: construction of matrix elements work in progress Talk by Emilis

Decomposition into scalar functions

- Consider process

$$\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2)\to\pi^a(p_1)\pi^b(p_2)$$

$$s=(q_1+q_2)^2=(p_1+p_2)^2 \qquad t=(q_1-p_1)^2=(q_2-p_2)^2 \qquad u=(q_1-p_2)^2=(q_2-p_1)^2$$

- Decompose amplitude into **scalar functions** Bardeen, Tung, Tarrach, see next talk for derivation

$$W_{\mu\nu}=\sum_{i=1}^{5}T^i_{\mu\nu}A_i$$

$$T_1^{\mu\nu}=q_1\cdot q_2 g^{\mu\nu}-q_2^\mu q_1^\nu \qquad T_2^{\mu\nu}=q_1^2q_2^2g^{\mu\nu}+q_1\cdot q_2 q_1^\mu q_2^\nu-q_1^2q_2^\mu q_2^\nu-q_2^2q_1^\mu q_1^\nu$$

$$T_3^{\mu\nu}=(t-u)(\tilde{T}_3^{\mu\nu}-\tilde{T}_4^{\mu\nu}) \qquad q_3=p_2-p_1$$

$$\tilde{T}_3^{\mu\nu}=q_1\cdot q_2 q_1^\mu q_3^\nu-q_1^2q_2^\mu q_3^\nu-\frac{1}{2}(t-u)q_1^2g^{\mu\nu}+\frac{1}{2}(t-u)q_1^\mu q_1^\nu$$

$$\tilde{T}_4^{\mu\nu}=q_1\cdot q_2 q_3^\mu q_2^\nu-q_2^2q_3^\mu q_1^\nu+\frac{1}{2}(t-u)q_2^2g^{\mu\nu}-\frac{1}{2}(t-u)q_2^\mu q_2^\nu$$

$$T_4^{\mu\nu}=q_1\cdot q_2 q_3^\mu q_3^\nu-\frac{1}{4}(t-u)^2g^{\mu\nu}+\frac{1}{2}(t-u)\left(q_3^\mu q_1^\nu-q_2^\mu q_3^\nu\right)$$

$$T_5^{\mu\nu}=q_1^2q_2^2q_3^\mu q_3^\nu+\frac{1}{2}(t-u)\left(q_1^2q_3^\mu q_2^\nu-q_2^2q_1^\mu q_3^\nu\right)-\frac{1}{4}(t-u)^2q_1^\mu q_2^\nu$$

- Matches onto **5 helicity amplitudes** $H_{\lambda_1\lambda_2}$, $\lambda_1\lambda_2\in\{++,+-,0+,+0,00\}$

Pion-pole contributions

- **Pion-pole contributions** defined by Cutkosky rules, this gives

$$A_1^\pi = -F_\pi^V(q_1^2)F_\pi^V(q_2^2)\left(\frac{1}{t-M_\pi^2}+\frac{1}{u-M_\pi^2}\right)$$

$$A_4^\pi = -F_\pi^V(q_1^2)F_\pi^V(q_2^2)\frac{2}{s-q_1^2-q_2^2}\left(\frac{1}{t-M_\pi^2}+\frac{1}{u-M_\pi^2}\right) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)\frac{2}{(t-M_\pi^2)(u-M_\pi^2)}$$

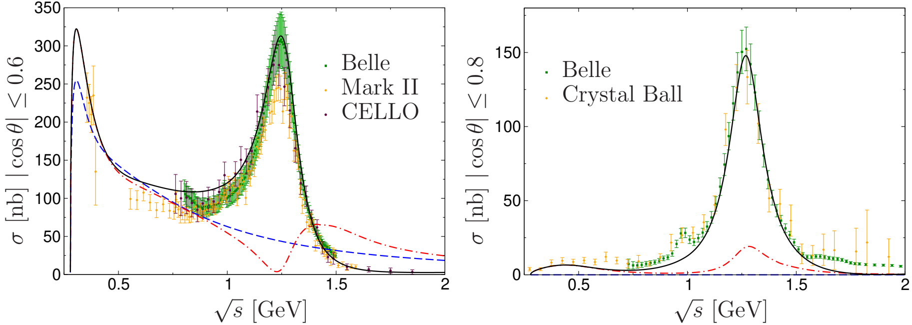
$$A_2^\pi = A_3^\pi = A_5^\pi = 0$$

- **Coincides with sQED times form factors**: F$\times$sQED $=$ FsQED in this case
- This is not guaranteed to happen, counterexample: nucleon pole vs. nucleon Born term in nucleon Compton scattering
- Result is easy to combine with multiloop techniques by writing

$$\frac{F_\pi^V(s)}{s} = \frac{1}{s}+\frac{1}{\pi}\int_{4M_\pi^2}^\infty ds'\,\frac{\mathrm{Im}\,F_\pi^V(s')}{s'(s'-s)} \qquad \text{or} \qquad \frac{F_\pi^V(s)}{s} = \frac{1}{s}\times\frac{1}{\pi}\int_{4M_\pi^2}^\infty ds'\,\frac{\mathrm{Im}\,F_\pi^V(s')}{s'-s}$$

- For $\gamma^*\gamma^*\to\pi\pi$ at low energies this is the dominant effect, but certainly for $\gamma^*\gamma^*\gamma\to\pi\pi$ we need to learn how to go beyond

Phenomenology of $\gamma\gamma \to \pi\pi$



- Pion pole dominant effect at low energies $\sqrt{s} \lesssim 1$ GeV

  $\hookrightarrow$ **unitarity corrections** from $\pi\pi$ S-wave rescattering $\Rightarrow$ ***f*$_0$(500) resonance**

- For $\sqrt{s} \gtrsim 1$ GeV, ***f*$_2$(1270) resonance**

  $\hookrightarrow$ D-wave rescattering of $3\pi \simeq \omega$, $2\pi \simeq \rho$ left-hand cuts

- Doubly-virtual process $\gamma^*\gamma^* \to \pi\pi$ (largely) predicted in terms of pion and vector-meson form factors

S-wave rescattering

- Pion-pole amplitude not unitary

  $\hookrightarrow$ **rescattering corrections** via $\pi\pi$ phase shifts $\delta_J(s)$

- Unitarity relation formulated at the level of **helicity amplitudes**

$$\operatorname{Im} h_J(s) = h_J(s) e^{-i\delta_J(s)} \sin \delta_J(s)$$

  $\hookrightarrow$ **inhomogenous Muskhelishvili–Omnès problem**

- S-wave solution reads

$$h_{0,++}(s) = N_{0,++}(s) + \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\sin \delta_0(s')}{|\Omega_0(s')|} \bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')} \bigg) N_{0,++}(s') + \frac{2q_1^2 q_2^2}{\lambda_{12}(s')} N_{0,00}(s') \bigg]$$

$$h_{0,00}(s) = N_{0,00}(s) + \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\sin \delta_0(s')}{|\Omega_0(s')|} \bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')} \bigg) N_{0,00}(s') + \frac{2}{\lambda_{12}(s')} N_{0,++}(s') \bigg]$$

$$\Omega_0(s) = \exp\bigg\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_0(s')}{s'(s'-s)} \bigg\} = 1 + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\operatorname{Im} \Omega_0(s')}{s'(s'-s)} = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\operatorname{Im} \Omega_0(s')}{s'-s}$$

- $N_{0,++}(s)$, $N_{0,00}(s)$: partial-wave projection of pion-pole terms

  $\hookrightarrow$ complicated dependence on q_i^2

First challenge: form of dispersive amplitudes

- $N_{0,++}(s)$, $N_{0,00}(s)$: **partial-wave projection** of pion-pole terms

$$N_{0,++}(s) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)\bigg[\frac{8}{\sigma_\pi(s)\lambda_{12}^{1/2}(s)}\bigg(\frac{sq_1^2q_2^2}{\lambda_{12}(s)}+M_\pi^2\bigg)Q_0(x_s)+2\frac{(q_1^2-q_2^2)^2-s(q_1^2+q_2^2)}{\lambda_{12}(s)}\bigg]$$

$$N_{0,00}(s) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)\frac{4}{\lambda_{12}(s)}\bigg[\frac{(q_1^2-q_2^2)^2-s^2}{\sigma_\pi(s)\lambda_{12}^{1/2}(s)}Q_0(x_s)+2s\bigg]$$

$$x_s = \frac{s-q_1^2-q_2^2}{\sigma_\pi(s)\lambda_{12}^{1/2}(s)} \qquad \sigma_\pi(s)=\sqrt{1-\frac{4M_\pi^2}{s}} \qquad Q_0(x)=\frac{1}{2}\int_{-1}^{1}\frac{dz}{x-z}$$

- Would like to have a **rational function** in the q_i^2 to incorporate into multiloop machinery, possible strategies:
  1. Revert partial-wave projection, perform z integral numerically
  2. Dispersion relation in q_i^2 possible?
  3. Wick rotation/Gegenbauer decomposition of box diagram?
  4. Dispersive calculation of box diagram?

First challenge: form of dispersive amplitudes

- Idea 1 produces

$$A_i(s) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)\frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^\infty ds'\,\frac{\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}\int_{-1}^1 dz\,\frac{\bar{A}_i(s',z)}{D(s',z)}$$

$$\bar{A}_1(s,z) = -2(s-q_1^2-q_2^2)s\lambda_{12}(s) + 2(s-4M_\pi^2)\Big[s(\lambda_{12}(s)+4q_1^2q_2^2) - ((q_1^2+q_2^2)\lambda_{12}(s)+12sq_1^2q_2^2)z^2\Big]$$

$$\bar{A}_2(s,z) = -4(s-4M_\pi^2)\Big[s(s-q_1^2-q_2^2) + (\lambda_{12}(s)-3s(s-q_1^2-q_2^2))z^2\Big]$$

$$D(s,z) = \lambda_{12}(s)\Big[s(\lambda_{12}(s)+4q_1^2q_2^2) - (s-4M_\pi^2)\lambda_{12}(s)z^2\Big] \qquad \lambda_{12}(s) = s^2+(q_1^2-q_2^2)^2-2s(q_1^2+q_2^2)$$

$\hookrightarrow$ rational function of q_i^2, but not quite of standard propagator form

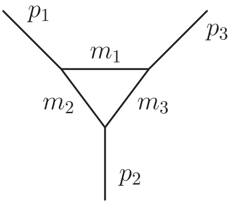
- Why care about rescattering corrections?

  $\hookrightarrow$ could be much more relevant for $\gamma^*\gamma^*\gamma\to\pi\pi$, due to ***P*-wave rescattering**

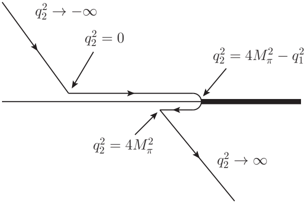
- One more challenge: **anomalous thresholds**

Second challenge: anomalous thresholds

- Consider the scalar loop function $C_0(s)$, $s = p_2^2$
- Fulfills the dispersion relation

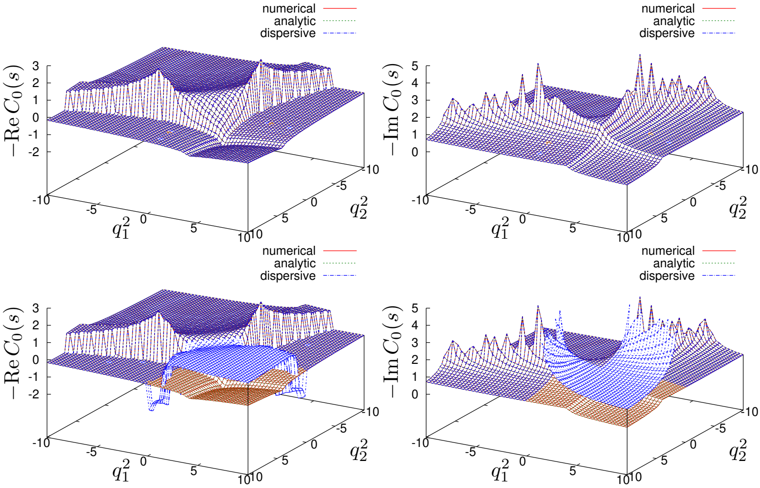


$$
\begin{aligned}
C_0(s) &= \frac{1}{2\pi i}\int\limits_{(m_2+m_3)^2}^{\infty} ds' \frac{\text{disc}\, C_0(s')}{s'-s} \\
&\quad + \theta\Big[m_3 p_1^2 + m_2 p_3^2 - (m_2+m_3)(m_1^2+m_2 m_3)\Big] \\
&\qquad \times \frac{1}{2\pi i}\int\limits_0^1 dx \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, C_0(s_x)}{s_x - s} \\
s_x &= x(m_2+m_3)^2 + (1-x)s_+ \\
s_+ &= p_1^2\frac{m_1^2+m_3^2}{2m_1^2} + p_3^2\frac{m_1^2+m_2^2}{2m_1^2} - \frac{p_1^2 p_3^2}{2m_1^2} - \frac{(m_1^2-m_2^2)(m_1^2-m_3^2)}{2m_1^2} \\
&\quad + \frac{1}{2m_1^2}\sqrt{\lambda(p_1^2,m_1^2,m_2^2)\lambda(p_3^2,m_1^2,m_3^2)}
\end{aligned}
$$



- Anomalous piece parameterizes the contour deformation from threshold to s_+

Second challenge: anomalous thresholds



- Example for $m_1 = m_2 = m_3 = M_\pi \equiv 1$ (upper: full, lower: without anomalous term)

  $\hookrightarrow$ this is exactly what we need here, **anomalous contribution** for $q_1^2 + q_2^2 \geq 4M_\pi^2$

Second challenge: anomalous thresholds

- In principle, we know how to deal with these anomalous contributions

$$h_0(s)\Big|_{\text{anom}} = \theta\big(q_1^2+q_2^2-4M_\pi^2\big)\,\frac{\Omega_0(s)}{2\pi i}\int_0^1 dx\,\frac{\partial s_x}{\partial x}\,\frac{\text{disc}_{\text{an}}\,h_0(s_x)}{s_x-s}$$

$$\text{disc}_{\text{an}}\,h_0(s) = -\frac{8\pi}{\sqrt{\lambda_{12}(s)}}\,\frac{t_0(s)}{\Omega_0(s)} \qquad s_x = 4M_\pi^2\,x + (1-x)s_+$$

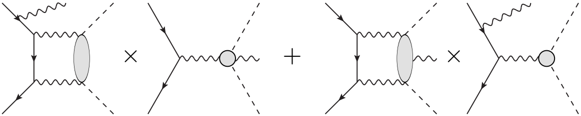
  but this makes it even harder to implement the result

- Ultimately, the anomalous contribution is related to the **singularities of the logarithm in the partial-wave projection**

$$Q_0(x_s) \propto \log\frac{s-q_1^2-q_2^2+\sigma_\pi(s)\lambda_{12}^{1/2}(s)}{s-q_1^2-q_2^2-\sigma_\pi(s)\lambda_{12}^{1/2}(s)}$$

- What happens with the anomalous contributions in Idea 1 above?
- Is there a way to account for the anomalous effects in a numerical partial-wave projection?

Conclusions



- **Pion-pole contributions** only the leading terms, corrections include
  - Rescattering effects
  - Higher left-hand cuts
- **Challenges in the implementation**
  - Functional form of dispersive amplitudes
  - Anomalous thresholds
- Example of ***S*-wave rescattering corrections**
  - Correspond to $f_0(500)$ resonance in $I = 0$
  - Might not be the most relevant ones phenomenologically, but easiest conceptually
  - Would suggest this as test case to learn on how to implement dispersive amplitudes beyond F$\times$sQED