Dispersive definition of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ for muon $g-2$ applications

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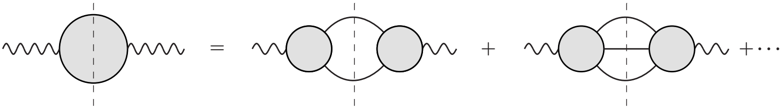
Overview

1. Role of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ in $(g-2)_\mu$
2. BTT: tensor decomposition for dispersive purposes
3. Soft-singular contribution to $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$

Overview

1. Role of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ in $(g-2)_\mu$
   - HVP: radiative return experiments
   - HLbL: inclusion of spin-2 resonances
2. BTT: tensor decomposition for dispersive purposes
3. Soft-singular contribution to $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$

More dispersive theory for HVP!

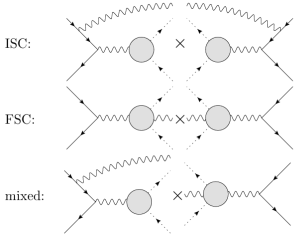


- **HVP in data-driven approach** - obtained from $\sigma(e^+e^- \to \text{hadrons}(+\gamma))$.
- **Radiative return experiments** - $e^+e^- \to \pi^+\pi^-\gamma$. Detection of hard $\gamma \to$ invariant mass of $\pi^+\pi^- \to$ scan energies without changing detector settings!
- **Challenges:**
  1. VFF measured from **initial-state radiation** contirbutions - **inferference** from **final-state radiation**.
  2. Experiment relies on theory input - so far FsQED $\to$ not model indep.!
- **More dispersive theory** could reduce uncertainties in **radiative corrections** introduced by model-dependence.

Radiative return at LO

Aliberti et. al SciPost Phys.Comm.Rep. 9 (2025)

- Computing $\sigma(e^+e^- \to \pi^+\pi^-\gamma)$ at LO - well-known ingredients:
  1. Pion VFF F_π^V;
  2. **Compton tensor** - crossed version of $\gamma^*\gamma^* \to \pi^+\pi^-$;
- Known dispersively to high precision $\Rightarrow$ see Martin Hoferichter's talk!



Radiative return at NLO

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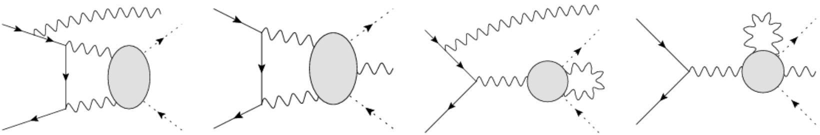
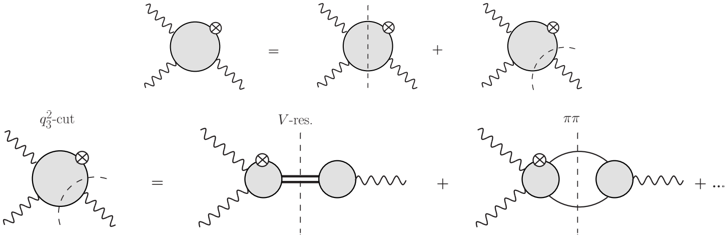


- NLO - new contributions arise:
  1. Well-known: Compton tensor (HLbL kin. or otherwise)
  2. In progress: rad. corrections to $F_\pi^V \Rightarrow$ Colangelo group
  3. New subprocesses: rad. corrections of Compton tensor and $\gamma^*\gamma^* \to \pi^+\pi^-\gamma$ - **no dispersive description yet!**
- For $\gamma^*\gamma^* \to \pi^+\pi^-\gamma$ - $\pi^+\pi^-$ in P-wave: ρ-res. + internal res. enchancement $\to$ FsQED not enough, unitarisation needed!
- 5-particle process in general kin. - dispersive approach very complicated. **Consider soft γ first!**

HLbL: triangle kinematics and tensor contributions

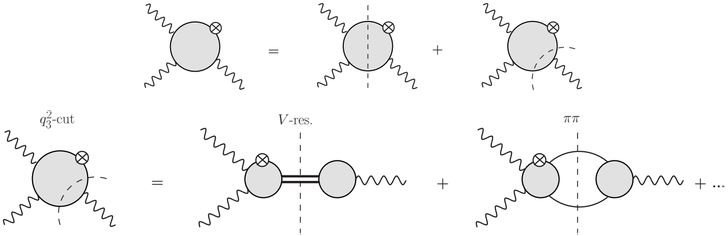
Lüdtke, Procura, Stoffer, JHEP 04 (2023) & JHEP 130 (2025)



- **HLbL:** large improvements from WP20, uncertainties dominated by axial and **tensor resonances**, matching to **short-distance constraints.**
- **Disp. relations in 4pt. kinematics:** spurious singularities in q_i^2 - so far unclear how to include spin-2 resonances in a consistent, model-indep. way!
- New **triangle kinematics** approach - take $q_4 \to 0$ immediately! Obtain HLbL from sum of two cuts.

HLbL: triangle kinematics and tensor contributions

Lüdtke, Procura, Stoffer, JHEP 04 (2023) & JHEP 130 (2025)



- Disperse in q_i^2 - **manifestly free of kin. singularities!**
- **Hefty price:** complicated unitarity cuts, new subprocesses $\gamma^*\gamma^*\gamma \to V$, $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$, $\pi\pi \to \gamma\pi\pi$.
- **Soft-singular parts** exist, cancel only in the sum of two triangle kin. cuts (up to finite remainder).
- **Soft-regular parts:** $\gamma^*\gamma^*\gamma \to \pi^+\pi$ includes tensor resonances through e.g. D-wave of $\pi\pi$ scattering via $\pi\pi \to \gamma\pi\pi$ subprocess. Can reconstruct dispersively!

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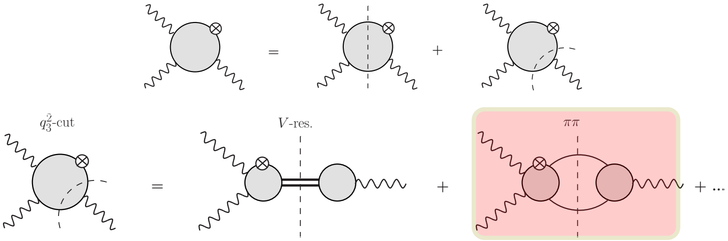

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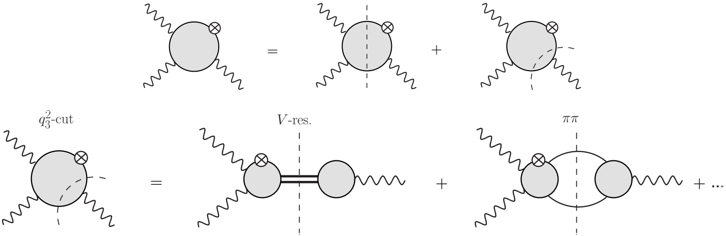

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Can construct pole and non-pole terms of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ for HLbL applications! Intuition for HVP!

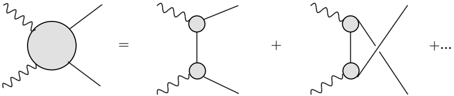
Overview

Motivation: BTT decomposition

Goal (Tensor basis for dispersive reconstruction)

Dispersive reconstruction of a scattering amplitude requires a tensor decomposition with:

- *No kin. singularities or zeroes;*
- ***Strongly desirable** - non-redundant basis;*



- Will discuss **BTT recipe**: *Bardeen, Tung, Phys. Rev. 173 5 (1968), Tarrach, Nuovo Cim A28 (1975)*. Satisfies the first requirement.
- **Complicated tensor structure of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$** : use $\gamma^*\gamma^* \to \pi^+\pi^-$ *Colangelo et al., JHEP 09 (2015)* to illustrate the procedure.
- Naïve decomposition - 10 tensor structures (q_i - photon momenta, p_j - pion):

$$W^{\mu\nu} = q_i^\mu q_j^\nu W_1^{ij} + g^{\mu\nu} W_2, \quad q_i \in \{q_1, q_2, p_1 - p_2\}.$$

- Ward identity - 5 linear relations between W_i - left with 5 indep, tensor structures. **Kinematic zeroes!**

BTT decomposition: the $\gamma^*\gamma^* \to \pi^+\pi^-$ subprocess (1)

Colangelo et al., JHEP 09 (2015)

1

Gauge projectors: apply $I^{\mu\nu} = g^{\mu\nu} - q_2^\nu q_1^\mu/(q_1 \cdot q_2)$ s.t. $W^{\mu\nu} = I^\mu_\lambda I^\nu_\sigma W^{\lambda\sigma}$. Now 5 strucs. W_i vanish, but now new tensor structures $T_i^{\mu\nu}$ have single and double poles in $(q_1 \cdot q_2)$.

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2

BT procedure: remove double poles by adding linear combinations of other tensor strucutres with non-singular coeffs; If no more can be removed - multiply strucs with $1/(q_1 \cdot q_2)^2$ poles by $(q_1 \cdot q_2)$ - repeat for single poles!

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3

Tarrach Redundancies: show that in $(q_1 \cdot q_2) \to 0$ basis $T_i^{\mu\nu}$ is **degenerate.** $\exists$ a kin. zero:

$$(q_1 \cdot q_3)(q_2 \cdot q_3) T_2^{\mu\nu} - q_2^2 (q_1 \cdot q_3) T_3^{\mu\nu} - q_1^2 (q_2 \cdot q_3) T_4^{\mu\nu} + q_1^2 q_2^2 T_5^{\mu\nu} = (q_1 \cdot q_2) T_6^{\mu\nu},$$

for some $T_6^{\mu\nu}$. **Add** $T_6^{\mu\nu}$ **to basis** - no kin. singularities or zeroes, but **basis redundant!**

BTT decomposition: the $\gamma^*\gamma^* \to \pi^+\pi^-$ subprocess (1)

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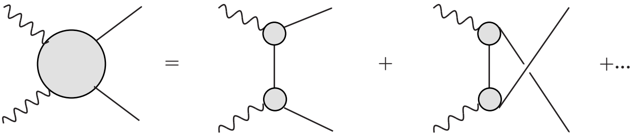
for some $T_6^{\mu\nu}$. **Add $T_6^{\mu\nu}$ to basis** - no kin. singularities or zeroes, but **basis redundant!**

4

Coincidentally for $\gamma^*\gamma^* \to \pi^+\pi^-$: bose symmetry for pions and photons - **additional zeroes**. Can arrive at a **non-redundant set:** 5 structures, usable for the dispersive description!

BTT decomposition: the $\gamma^*\gamma^* \to \pi^+\pi^-$ subprocess (2)

Colangelo et al., JHEP 09 (2015)



- Resultant tensor decomposition (A_i - can be described dispersively):

$$W^{\mu\nu} = \sum_{i=1}^{5} A_i T_i^{\mu\nu}.$$

- $5 \times A_i$ structures - matches # **of unique helicity amplitudes**. 1:1 correspondence useful for **unitarisation** (inclusion of final-state rescattering effects).
- **Redundant basis:** amplitude invariant under **Tarrach shift** (Δ - arbitrary non-singular function).

$$W^{\mu\nu} = \sum_{i=1}^{6} B_i \tilde{T}_i^{\mu\nu}, \qquad \delta W^{\mu\nu} = 0 \text{ if}$$

$$B_1' = B_1, \qquad B_2' = B_2 - \frac{(q_1 \cdot q_3)(q_2 \cdot q_3)}{(q_1 \cdot q_2)}\Delta \qquad B_3' = B_3 + \frac{q_2^2(q_1 \cdot q_3)}{(q_1 \cdot q_2)}\Delta,$$

$$B_4' = B_4 + \frac{q_1^2(q_2 \cdot q_3)}{(q_1 \cdot q_2)}\Delta, \qquad B_5' = B_5 - \frac{q_1^2 q_2^2}{(q_1 \cdot q_2)}\Delta, \qquad B_6' = B_6 + \Delta.$$

- Tarrach shifts will be crucial in constructing the pion pole contribution to $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$.

BTT decomposition for 3 photons: general kinematics

Lüdtke, Procura, Stoffer, JHEP 04 (2023)

$$\langle \pi^+(p_1)\,\pi^-(p_2)\,|\gamma^*(q_1,\lambda_1)\,\gamma^*(q_2,\lambda_2)\,\gamma(q_3,\lambda_3)\rangle$$
$$=: ie^3(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2-q_3)\,\epsilon_\mu^{\lambda_1}(q_1)\,\epsilon_\nu^{\lambda_2}(q_2)\,\epsilon_\lambda^{\lambda_3}(q_3)\,\mathcal{M}^{\mu\nu\lambda}(p_1,p_2,q_1,q_2)$$

- The BTT for $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ - non-trivial, results in a **redundant basis of 74 tensor structures** in general 5-pt. kinematics:
$$\mathcal{M}^{\mu\nu\lambda} = \sum_{i=1}^{74} T_i^{\mu\nu\lambda} C_i.$$
- **38 Tarrach redundancies** in D-dim.!
- Pick 36 tensors T_i in D-dim. for projection. Resultant $36\times C_i$ **contain many kinematic singularities** of the form $1/(q_j \cdot q_k)$, **removed using Tarrach redundancies 'by hand'.**

Problem: Soft-singular pion pole - only meaningful in 5-pt. kinematics.

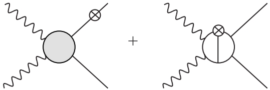
1. If direct construction $\mathcal{M}^{\mu\nu\lambda}_{\text{pole}}$ possible - bypasses kin. singularities.
2. Otherwise, work at C_i level - need a scheme for singularity cancellation using Δ_i's.

BTT decomposition for 3 photons: soft-γ limit

Lüdtke, Procura, Stoffer, JHEP 04 (2023)

- $q_3 \to 0$ limit: Assume a **gauge-invariant separation:**

$$C_i = C_i^{\text{poles}} + C_i^{\text{non-pole}} =$$



- Once C_i^{poles} are known, soft-regular amplitude part in triangle kinematics:

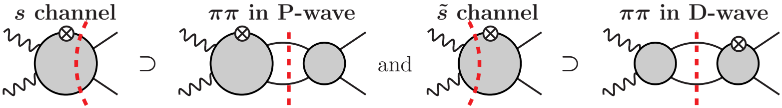
$$\mathcal{M}_{\text{reg}}^{\mu\nu\lambda} = \sum_{i=1}^{74} \hat{T}_i^{\mu} C_i^{\text{non-pole}}, \; \frac{\partial}{\partial q_3^{\sigma}} \mathcal{M}_{\text{reg}}^{\mu\nu\lambda}\bigg|_{q_3=0} = \sum_{i=1}^{74} \left(\frac{\partial}{\partial q_3^{\sigma}} \hat{T}_i^{\mu\nu\lambda} \right)\bigg|_{q_3=0} C_i^{\text{non-pole}}(q_3 = 0)\,,$$

  which allows to extract $C_i^{\text{non-pole}}$ (can be described dispersively). Can show that **only 27 structures survive** - **matches # of helicity amplitudes!**

- For projection: directly construct a rank-4 basis of 27 strucutres in $q_3 \to 0$ limit $T_i^{\mu\nu\lambda\sigma}$ - **remnants of Ward identity** wrt. q_3 imposed as **antisymmetry** in $\lambda \leftrightarrow \sigma$,

$C_i^{\text{non-pole}}$ will contain spin-2 resonance effects in HLbL!

Aside: obtaining the soft-regular contribution



- **Non-pole part**: two types of contributions to $\pi\pi$ cut (other intermediates not included) - set up inhom. Omnès problem.
- $\tilde{s}-$**channel:** $\pi\pi \to \gamma\pi\pi \Rightarrow$ Luedke, Niklas-Toelstede, Stoffer, in preparation.
- From $\pi\pi \to \pi\pi\gamma$: try **reconstruction theorem** approach. Partial waves related to set of functions obeying simple disp. relations, integral system solved numerically!
- **Subtraction constants:** constrain their number by imposing asymp. scaling, the rest obtain from $SU(2)$ χPT.

Current work: replace $\pi\pi$ in D-wave with $f_2(1270)$ res. in NWA.

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   - Motivation and Goals
   - Poles of $\pi^0\pi^0 \to \gamma^*\pi^+\pi^-$: direct construction
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   - An *intermezzo*: sQED example
   - Poles of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$: systematic construction

Motivation and goals

Goal (Dispersive soft-pole construction)

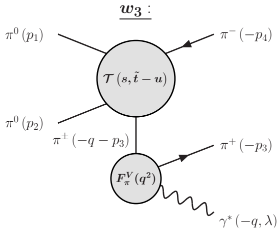
Construct an object that captures all soft singularities of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ *in terms of* $\gamma^*\gamma^* \to \pi^+\pi^-$ *and VFF with:*

1. *Unique* π*-pole im. parts;*
2. *No kin. singularities;*
3. *Symmetries: gauge invariance, crossing;*

- **Why dispersive?** Alternatives (Low's thrm., Adler-Dothan) - only q_1^{-1}, q_1^0-terms in soft-q_1 expansion: remainder may be singular - **spoils disp. description of** $\boldsymbol{C_i^{\text{non-pole}}}$.
- **Complicated tensor structure of** $\boldsymbol{\gamma^*\gamma^*\gamma \to \pi^+\pi^-}$: 74 **BTT,** 38 Tarrach redundancies Δ_i - find simpler example first.
- **Starting point:** 5-particle subprocess $\pi^0\pi^0 \to \gamma^*\pi^+\pi^-$ (*J. Lüdtke, PhD thesis*) - use as analogy today.

Important: result will **not be unique!** Any real part wrt. π-pole kinematic invariants (satisfying 1-3) works.

Poles of $\pi^0\pi^0 \to \gamma^*\pi^+\pi^-$

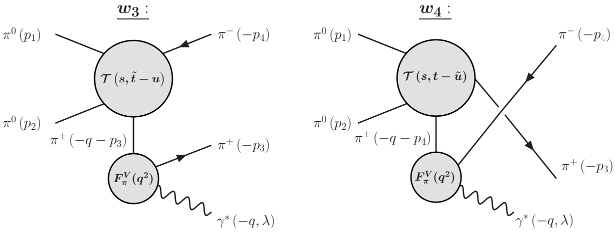


- Pion-pole im. part:

$$\mathrm{Im}^{\pi}_{w_3}\mathcal{M}^\mu = -\pi\,(2p_3+q)^\mu\,\mathcal{T}(s,\tilde{t}-u)F_\pi^V(q^2)\delta((p_3+q)^2-M_\pi^2),$$
$$\mathrm{Im}^{\pi}_{w_4}\mathcal{M}^\mu = +\pi\,(2p_4+q)^\mu\,\mathcal{T}(s,t-\tilde{u})F_\pi^V(q^2)\delta((p_4+q)^2-M_\pi^2),$$

- **Amplitude ansatz?**

Poles of $\pi^0\pi^0 \to \gamma^*\pi^+\pi^-$



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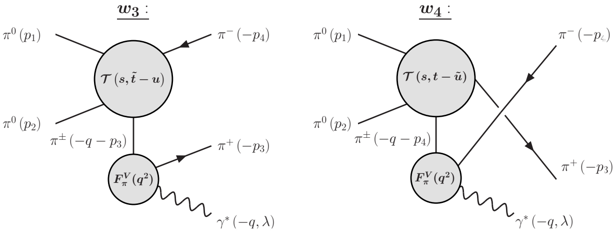
- **Amplitude ansatz?**

$$\mathcal{M}^\mu \overset{???}{=} F_\pi^V(q^2)\left(\frac{(2p_3+q)^\mu}{(p_3+q)^2-M_\pi^2}\mathcal{T}(s,\tilde{t}-u)-\frac{(2p_4+q)^\mu}{(p_4+q)^2-M_\pi^2}\mathcal{T}(s,t-\tilde{u})\right).$$

$\Rightarrow$ Symmetries? Gauge invariance?

Poles of $\pi^0\pi^0 \to \gamma^*\pi^+\pi^-$

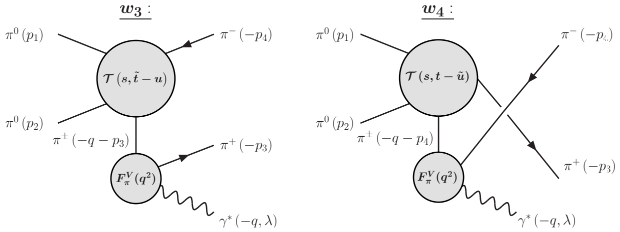


- Not gauge invariant! Additional term:

$$\mathcal{M}^\mu = F_\pi^V(q^2)\left(\frac{(2p_3+q)^\mu}{(p_3+q)^2-M_\pi^2}\mathcal{T}(s,\tilde{t}-u) - \frac{(2p_4+q)^\mu}{(p_4+q)^2-M_\pi^2}\mathcal{T}(s,t-\tilde{u}) \boldsymbol{-2\,(p_1-p_2)^\mu\,\Delta\mathcal{T}}\right),$$

with $\Delta\mathcal{T} := \left(\mathcal{T}(s,\tilde{t}-u)-\mathcal{T}(s,t-\tilde{u})\right)/(\tilde{t}-u-t+\tilde{u})$. **Real, not singular!**

Poles of $\pi^0\pi^0 \to \gamma^*\pi^+\pi^-$



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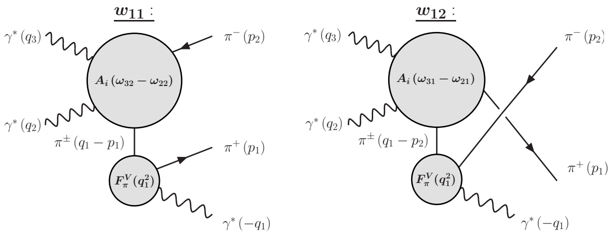
- For $3\gamma \to 2\pi$: $\mathcal{T} \overset{???}{\to} A_i \times T_i^{\nu\lambda}$

Try similar approach for $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ pole pieces

Poles of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$: direct construction



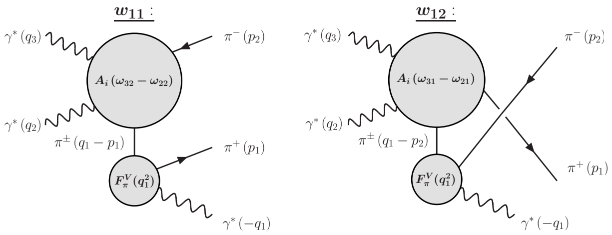
- Want result wrt VFF and $\gamma^*\gamma^* \to \pi^+\pi^-$.
- **Right imaginary parts** (note shift in q_1):

$$\mathcal{M}^{\mu\nu\lambda} = F^V_\pi(q_1^2)\left[\frac{(2p_1-q_1)^\mu}{(p_1-q_1)^2-M_\pi^2}W^{\nu\lambda}(p_1-q_1,p_2,q_2) - \frac{(2p_2-q_1)^\mu}{(p_2-q_1)^2-M_\pi^2}W^{\nu\lambda}(p_2-q_1,p_1,q_2) + \boldsymbol{R^{\mu\nu\lambda}}\right]$$

where $R^{\mu\nu\lambda}$ - **real, finite, very hard to construct.**

- $W^{\mu\nu} = \sum_{i=1}^{5} T_i^{\mu\nu} A_i$ - split into 5 parts.

Poles of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$: direct construction

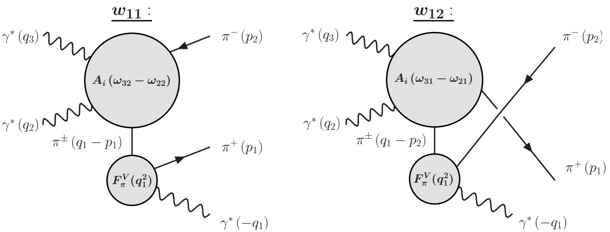


- First 2 functions $A_i, i = 1, 2$: $T_i^{\nu\lambda}$ indep. of soft-q_1 shift. Simplification:

$$\mathcal{M}_{A_i}^{\mu\nu\lambda} = F_\pi^V(q_1^2) T_i^{\nu\lambda} \left[\frac{(2p_1 - q_1)^\mu}{(p_1 - q_1)^2 - M_\pi^2} A_i^{11} - \frac{(2p_2 - q_1)^\mu}{(p_2 - q_1)^2 - M_\pi^2} A_i^{12} + R^\mu \right],$$

assumed $\boldsymbol{R^{\mu\nu\lambda} = T_i^{\nu\lambda} \times R^\mu}$. Does it work?

Poles of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$: direct construction



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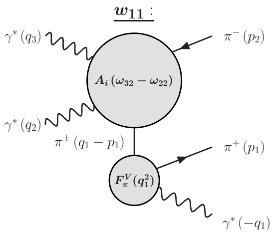
- **Yes!**

$$\mathcal{M}_{A_i}^{\mu\nu\lambda} = F_\pi^V(q_1^2)T_i^{\nu\lambda}\left[\frac{(2p_1-q_1)^\mu}{(p_1-q_1)^2-M_\pi^2}A_i^{11} - \frac{(2p_2-q_1)^\mu}{(p_2-q_1)^2-M_\pi^2}A_i^{12} - 2(q_2-q_3)^\mu\,\frac{A_i^{11}-A_i^{12}}{W_1}\right].$$

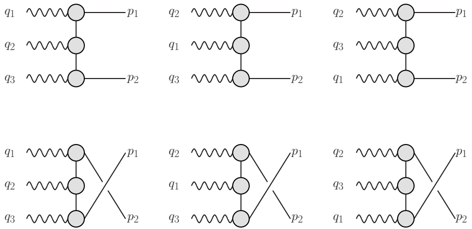
Poles of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$: direct construction



- **Other functions** A_i, $i = 3, 4, 5$: soft-q_1 shift - $T_i^{\nu\lambda}$ differs in two channels.
- **Can prove:** Gauge inv. + pion crossing $\Rightarrow$ simple factorization $R^{\mu\nu\lambda} \sim T_i^{\nu\lambda} \times R^\mu$ **fails.**
- Can't construct $\mathcal{M}^{\mu\nu\lambda}$ directly $\Rightarrow$ will need to deal with many Δ_i in scalar functions C_i.

Need a more systematic approach!

An *intermezzo*: sQED example (1)



- **sQED tree-level amplitude:** gain intuition on redundancy cancellation. F×sQED gives **pole-pole contribution** - see *Colangelo, Hoferichter, Stoffer JHEP 09 (2015).*
- 15 out of 74 $C_i \neq 0$ after projection (all contain kin. singularities):

$$C_i^{\text{sQED}} \neq 0, \quad i \in \{1-4, 8-10, 20-22, 26-28, 32, 35\}.$$

- However, the redundancy structure is complicated - **any guiding principle?**

$$\begin{aligned}\Delta C_{22}^{\text{sQED}} = &-\frac{(q_1\cdot q_4)(q_2\cdot q_4)(q_2\cdot q_5)(q_3\cdot q_4)(q_3\cdot q_5)}{2(q_1\cdot q_2)(q_1\cdot q_3)(q_2\cdot q_3)}\Delta_1 - \frac{(q_2\cdot q_5)(q_3\cdot q_5)}{(q_2\cdot q_3)}\Delta_4 \\ &+ \frac{(q_2\cdot q_5)(q_3\cdot q_4)}{(q_1\cdot q_2)}\Delta_5 - \frac{(q_2\cdot q_4)(q_3\cdot q_4)}{(q_2\cdot q_3)}\Delta_{10} - \frac{(q_2\cdot q_5)(q_3\cdot q_4)(q_3\cdot q_5)}{2(q_1\cdot q_3)(q_2\cdot q_3)}\Delta_{16} \\ &+ \frac{(q_2\cdot q_4)(q_2\cdot q_5)(q_3\cdot q_4)}{2(q_1\cdot q_2)(q_2\cdot q_3)}\Delta_{19} + \frac{(q_1\cdot q_4)(q_2\cdot q_5)(q_3\cdot q_4)}{2(q_1\cdot q_2)(q_1\cdot q_3)}\Delta_{20} + \frac{(q_2\cdot q_5)(q_3\cdot q_4)}{2(q_1\cdot q_3)}\Delta_{24} + \dots\end{aligned}$$

An *intermezzo*: sQED example (2)

Dramatically simplifying principle: *if $C_i = 0$ after the projection, all Tarrach redundancies contributing to it will be set to zero.* Extremely useful for picking Δ_i in the dispersive pole construction.

- **In sQED:** 6 non-zero redundancies Δ_{2-4}, Δ_{33-35} - enough to remove all kin. singularities in C_i^{sQED}. Affect disjoint subsets of scalar functions (*redundancy families*):

$$\{\Delta_{2,3,4} : C_{20-22}, C_{26-28}, C_{32}, C_{35}\}, \qquad \{\Delta_{35} : C_{2,8}\}, \qquad \{\Delta_{34} : C_{3,9}\}, \qquad \{\Delta_{33} : C_{4,10}\}.$$

- Simple final result, e.g.:

$$C_2^{\text{sQED}} = \left(\frac{1}{\omega_{11}} + \frac{1}{\omega_{12}}\right)\left(\frac{1}{\omega_{22}\omega_{31}} - \frac{1}{\omega_{21}\omega_{32}}\right),$$

$$C_8^{\text{sQED}} = -\left(\frac{1}{\omega_{11}\omega_{12}\omega_{21}} + \frac{1}{\omega_{11}\omega_{12}\omega_{22}} + \frac{1}{\omega_{11}\omega_{12}\omega_{31}} + \frac{1}{\omega_{11}\omega_{12}\omega_{32}} + \frac{1}{\omega_{11}\omega_{21}\omega_{32}} + \frac{1}{\omega_{12}\omega_{22}\omega_{31}}\right),$$

$$\Delta_{35}^{\text{sQED}} = \frac{2(q_2 \cdot q_3)}{\omega_{11}\omega_{12}}\left(\frac{1}{\omega_{22}\omega_{31}} + \frac{1}{\omega_{21}\omega_{32}}\right), \qquad \text{with} \qquad \omega_{ij} = (q_i - p_j)^2 - M_\pi^2.$$

- Results above - can use to form the **F×sQED part of the general soft-pole contribution.**

Poles of $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$: systematic construction

Inspired by: J. Lüdtke, PhD thesis

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Singular $\boldsymbol{\mathcal{M}^{\mu\nu\lambda}}$ construction: Project $\mathrm{Im}\mathcal{M}^{\mu\nu\lambda}$ onto $\mathrm{Im}C_i$ (no gauge-inv., symmetry issues). Disp. integral: get singular C_i, but right im. parts. **Singularities: only in $\boldsymbol{\mathrm{Re}\mathcal{M}^{\mu\nu\lambda}}$.**

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Tarrach $\boldsymbol{\Delta_i}$: write schematically $\mathrm{Im}C_i \supset \sum_j (q_a \cdot q_b)/(q_c \cdot q_d)\Delta_j = \sum_j \mathrm{Im}\delta C_i^j$ - construct additional real parts $\mathrm{Re}\delta\mathcal{M}_j^{\mu\nu\lambda} \Rightarrow$ **use to cancel kin. singularities.**

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Problems: $\#$ of singularities < 38, Δ_i not unique $\Rightarrow$ **not a linear algebra problem.**
sQED toy: simplifying principles, some $\Delta_i \to 0$, **but not enough to specify uniquely**.

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Group Δ_i, C_i in isolated subsets, cancel $1/(q_i \cdot q_j)^n$ with Δ_i sQED-inspired ansatzes. Each Tarrach real part $\mathrm{Re}\delta\mathcal{M}_j^{\mu\nu\lambda}$ - **multiple singularities** if e.g. we find Δ_i that cancels $1/(q_1 \cdot q_2)$, make sure that $1/(q_2 \cdot q_3)$-piece also finite. **Rather tedious.**

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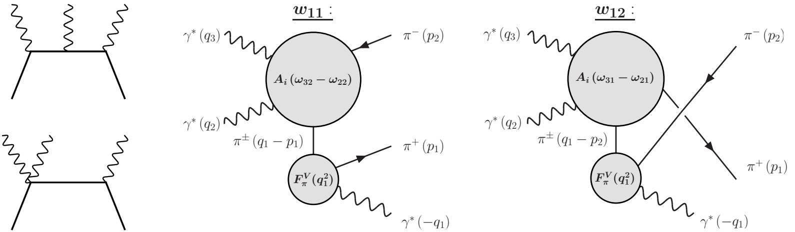
Status: pole pieces fully computed, tested with $SU(2)$ χPT at NLO.

F×sQED vs dispersive construction



(a) (b)

- How does LO F×sQED (a) compare with our (appropriately-crossed) dispersive result (b), with only π-pole of $\gamma^*\gamma^* \to \pi^+\pi^-$?
- **Difference is purely real** in all 6 π-pole channels!
- **Beyond pole-pole:** need full Compton tensor f-ions A_i, non-pole pieces.
- **Note:** **our result - not unique!** Could have disp. pole $\equiv$ F×sQED, but have no control over it during construction.

Therefore, F×sQED **is a reasonable starting point** for $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ evaluation!
However, with $\pi^+\pi^-$ in P-wave, hadronic resonance effects may be significant.

Conclusions

- $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ subprocess - may shine light into reducing uncertainties in **both HVP and HLbL contributions.**
- Current focus on the soft-γ limit - **spin-2 contributions to HLbL.** General case for HVP - future goal.
- Constructed the **soft-singular pion-pole part of** $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$ - **shown that F$\times$sQED is a valid starting point for the theory input at radiative-return experiments.**
- Hadronic resonances may be a relevant effect - effects in soft-γ limit under investigation.

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Thank you for your attention!