Evaluation of two-loop Feynman integrals for $e^+e^- \longrightarrow \gamma\gamma^*$

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Theory and Phenomenology of Fundamental Interactions
UNIVERSITY AND INFN · BOLOGNA

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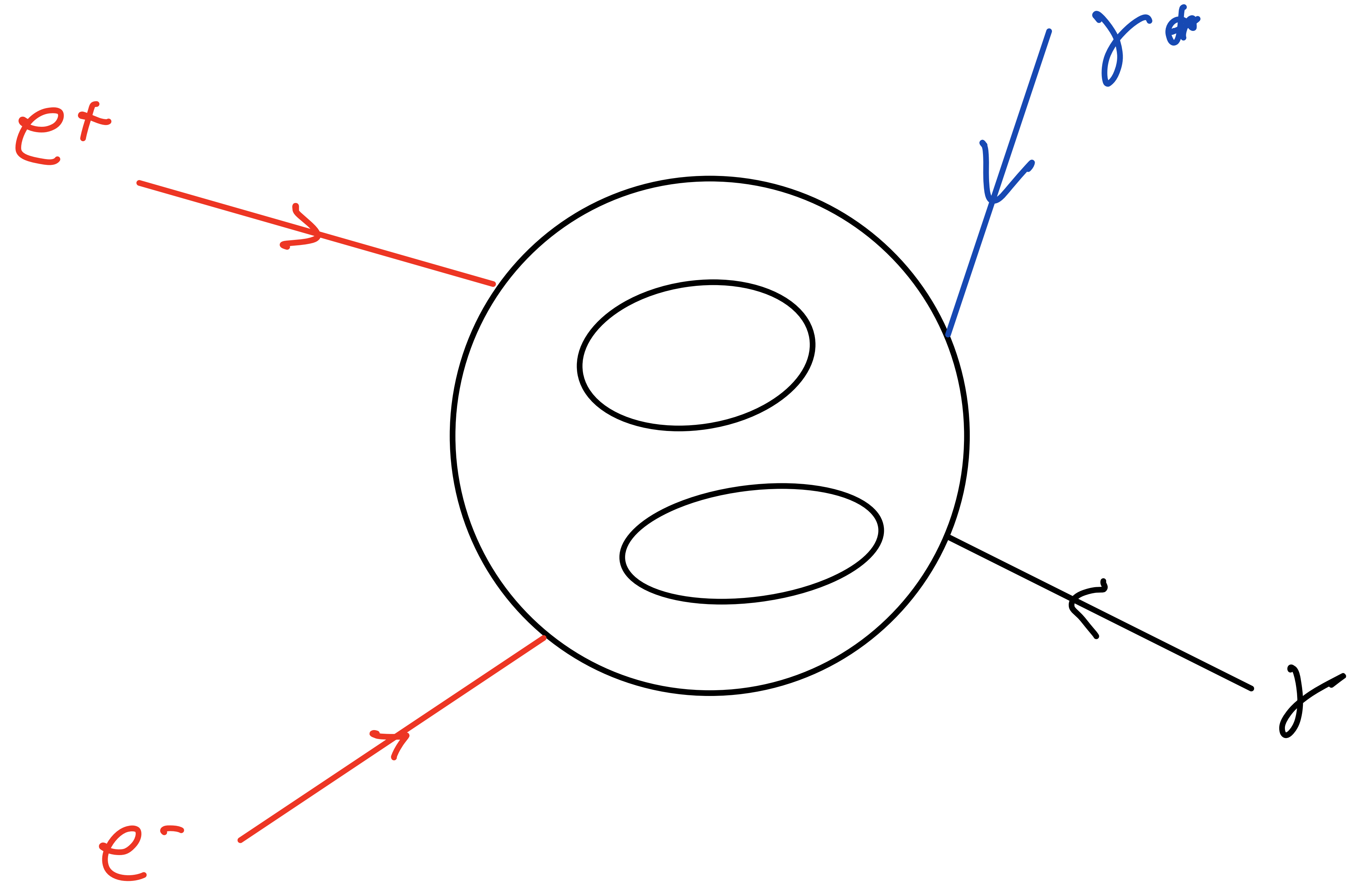
Setup and kinematics

$e^+(p_1) + e^-(p_2) + \gamma(p_3) + \gamma^*(p_4) \to 0$

- We want to compute the 2-loop amplitude for $e^+e^- \to \gamma\gamma^*$ with massive electrons
- $p_1^2 = p_2^2 = m^2, \quad p_3^2 = 0, \quad p_4^2 = q^2$
- 4 variables: $\vec{x} := \{s, t, m^2, q^2\}$
- We need to compute Feynman integrals in dimensional regularisation: $d = 4 - 2\varepsilon$



Integral families

$$G_{a_1,\ldots,a_9} = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \frac{1}{\boxed{D_1^{a_1}\cdots D_9^{a_9}}}, \qquad (a_1,\ldots,a_9) \in \mathbb{Z}^9$$

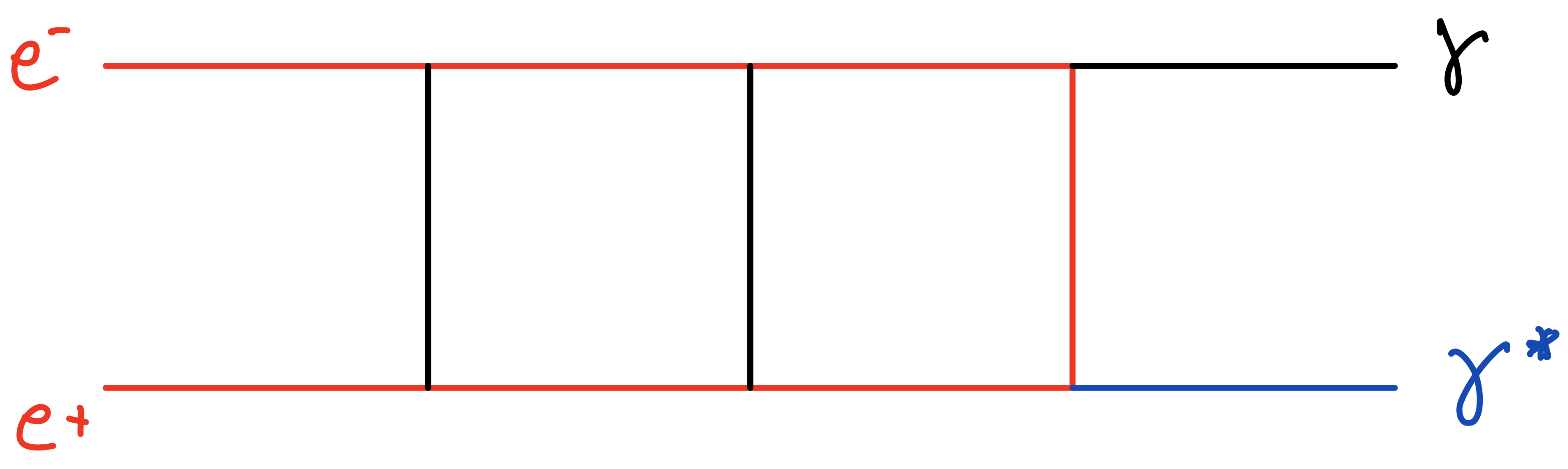
Inverse propagators of Feynman integrals

$$D_j = l_j^2 - m_j^2$$

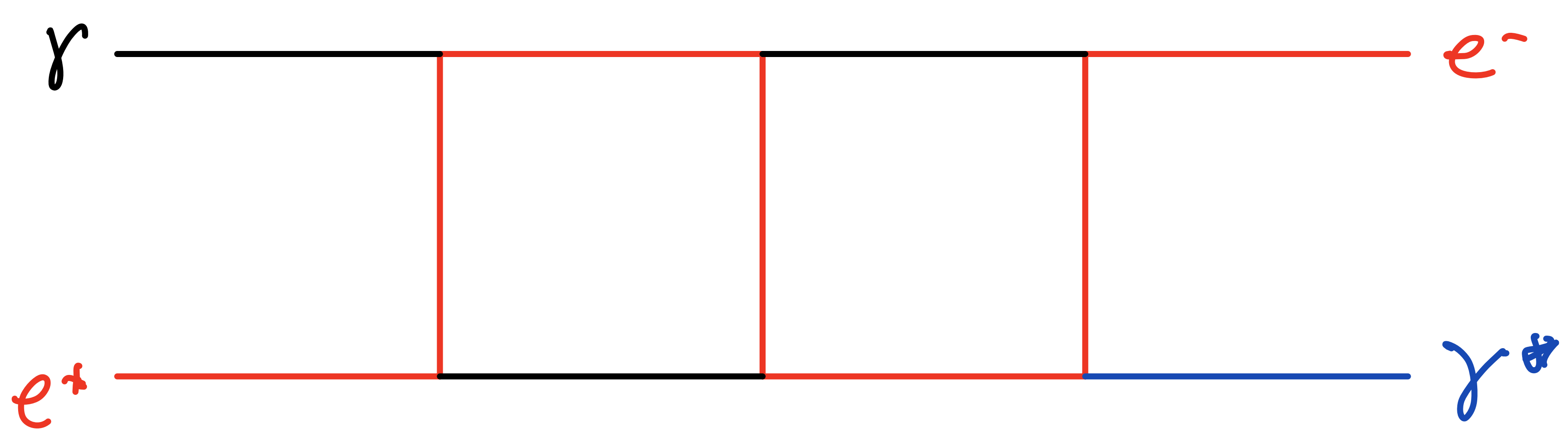
- Integrals grouped in sectors by their set of denominators
- Amplitudes: all scalar products $k_i \cdot p_j$ and $k_i \cdot k_j$ expressed in terms of denominators

$\Longrightarrow$ Beyond one-loop we need irreducible scalar products (ISPs). Here: 2 ISPs

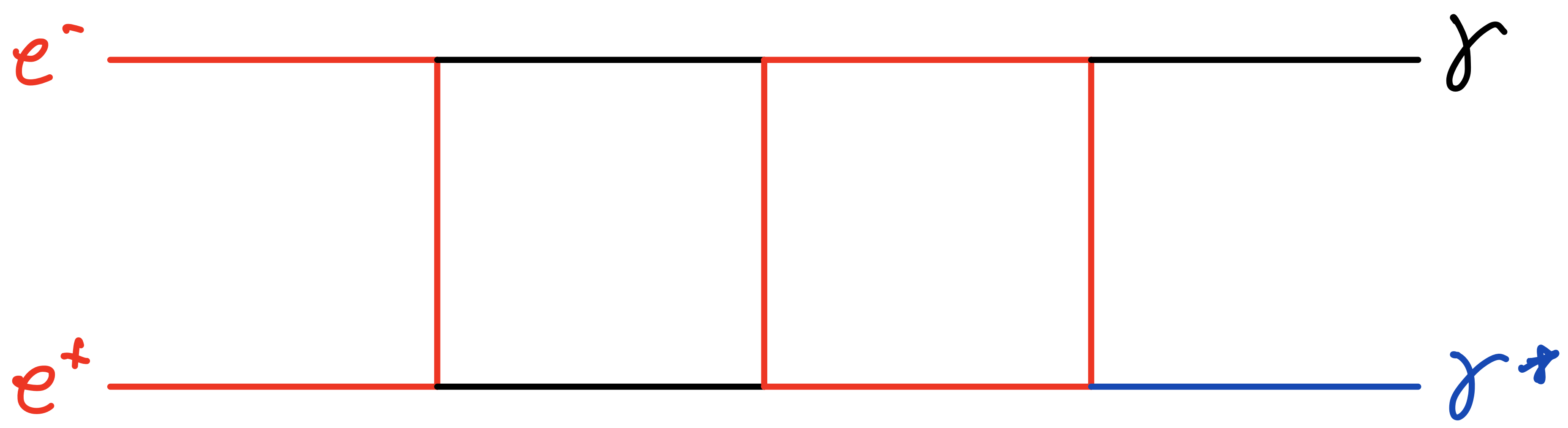
Our integral families



PL_1



PL_2



PL_3

+ non-planars

Master integrals and differential equations

- Integral families have a finite basis of master integrals (MIs)
- MIs satisfy linear differential equations (DEs)

$$\partial_\xi \vec{I}(\vec{x};\varepsilon) = B_\xi(\vec{x};\varepsilon)\cdot\vec{I}(\vec{x};\varepsilon)$$

- Eventually, we want to solve the DEs fully numerically $\Longrightarrow$ fast evaluation, can be implemented in a Monte Carlo generator
- Complexity of the DEs depends on the basis

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Basis choice is key part of the calculations!

Structure of the DEs

- In general, the matrices $B_\xi(\vec{x};\varepsilon)$
  - Contain only rational functions
  - Mix ε and the kinematics
  - Have spurios poles
- General strategy
  1. Select "good" starting basis $\vec{I}$
  2. Gauge transformation $\vec{I} \to T \cdot \vec{I}$

$$B_\xi \to \left(T \cdot B_\xi + \partial_\xi T\right) \cdot T^{-1}$$

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Makes the DEs "nicer", but introduces new functions

Optimal (?) basis

[Henn 2014]: DEs in canonical form (no general algorithm)

$$\mathrm{d}\vec{I}(\vec{x};\varepsilon) = \varepsilon \boxed{\mathrm{d}\tilde{A}(\vec{x})} \vec{I}(\vec{x};\varepsilon)$$

one-forms with at most simple poles

- No spurious poles
- ε-dependence factorises: solution at each order depends only on previous order

▸ In the best understood cases the one-forms are logarithmic

$$\mathrm{d}\tilde{A}(\vec{x}) = \sum_i a^{(i)} \,\mathrm{d}\log \boxed{W_i(\vec{x})}$$

Letters: algebraic functions

Integrand analysis

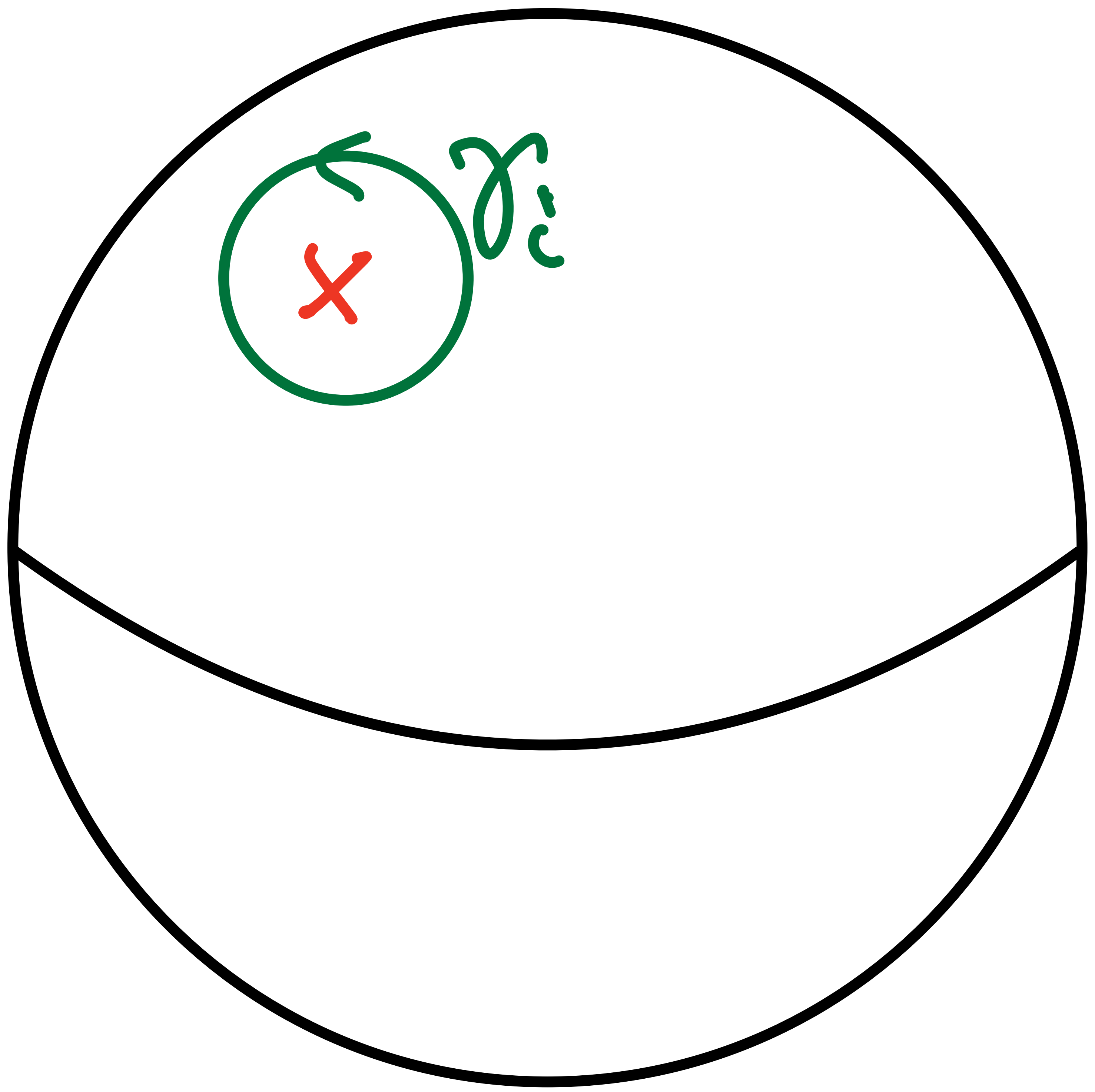
- The type of functions that appear in the solution and in T depend on the differential forms of my integrand:

$$I \sim \int \mathrm{d}f_1(\vec{z}) \wedge \ldots \wedge \mathrm{d}f_2(\vec{z})$$

- Connection to geometry: one-forms are defined over some manifold
- We can easily analyse the geometry of an integral in the Baikov representation

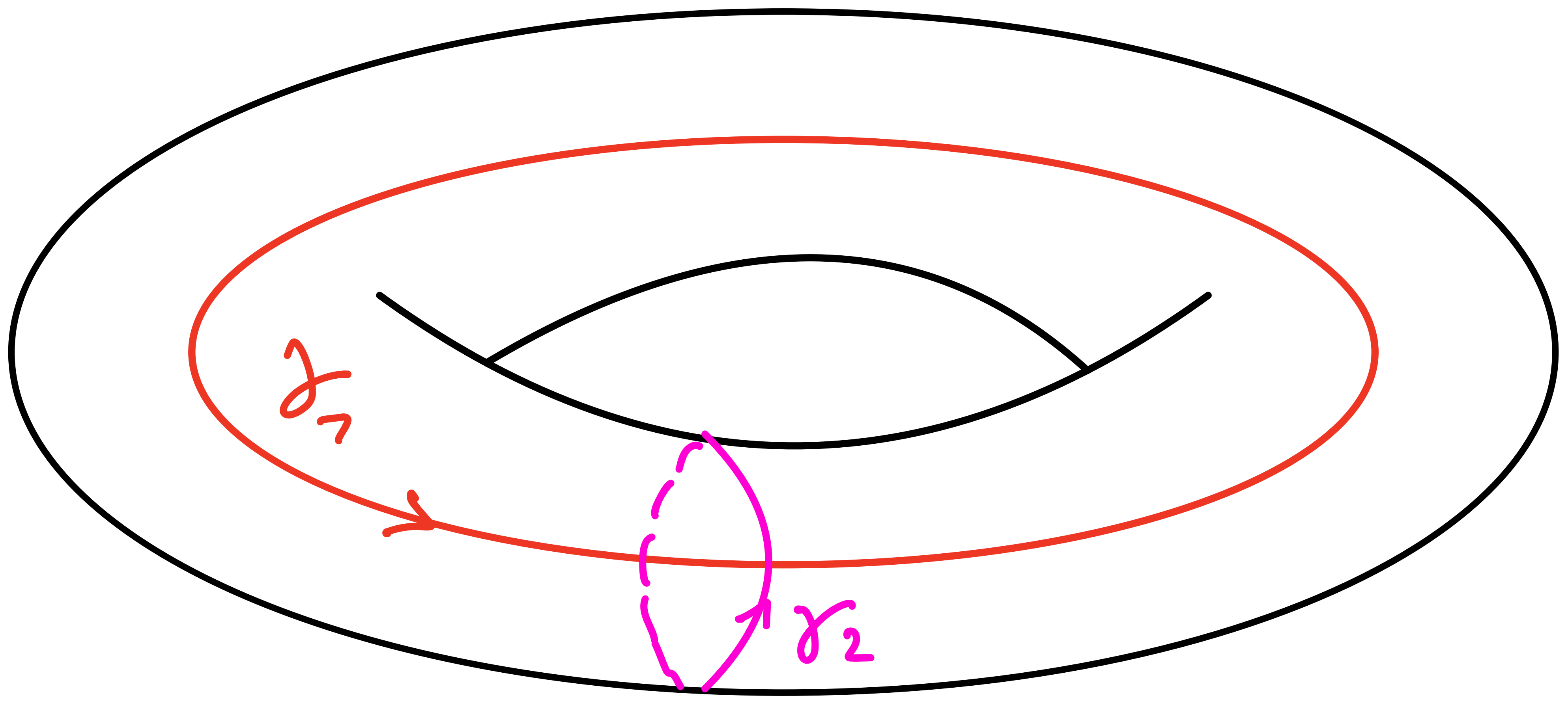
Logarithmic case is well understood

- One-loop problems, massless propagators
- Geometry: punctured Riemann sphere
- Starting basis linear in ε
- Canonical basis through gauge transformation T involving only rational functions and simple square roots



Elliptic case

- Massive propagators
- Geometry: torus
- Starting basis:
  - Quadratic in ε relatively easy to get
  - Conjecture: algebraic basis linear in ε [Chaubey, Sotnikov 2025]?
- Canonical basis: matrix T involves transcendental functions



How we deal with elliptic integrals

Goal: obtain a good basis without introducing transcendental functions

- Simple ε-dependence
  - No poles
  - Maximum degree as low as possible (2 in this case)
- We choose finite elliptic MIs
  - Poles of the amplitude do not contain elliptic functions
  - Allows us to apply dlog techniques up to the last order in the ε-expansion

Representation of the DEs

- The entries that do not involve elliptic integrals are ε-factorised and contain only logarithmic one-forms
- The DEs are free of spurious poles
- Compact representation:

$$\mathrm{d}\vec{I}(\vec{x};\varepsilon) = \mathrm{d}A^{(F)}(\vec{x};\varepsilon)\cdot\vec{I}(\vec{x};\varepsilon), \qquad \mathrm{d}A^{(F)}(\vec{x};\varepsilon) = \sum_{k=0}^{2}\varepsilon^k\Big[\sum_\alpha c^{(F)}_{k\alpha}\mathrm{d}\log(W_\alpha(\vec{x})) + \sum_\beta d^{(F)}_{k\beta}\omega_\beta(\vec{x})\Big]$$

- The one-forms ω_β
  - Are linearly independent
  - Are chosen in such a way that the polynomial degrees are minimised

Evaluation strategy: short term

- Evaluation through generalised series expansion method
  - Easy to set up: requires boundary point (e.g. `AMFlow` [Liu, Ma 2022]) and DEs
  - Sensitive to singularities of DEs $\Longrightarrow$ “safer” than fully numerical approach
- Performance
  - `DiffExp` [Hidding 2020]: order few minutes based on experience
  - `AMFlow` DE solver: factor 10 gain for other processes
  - `LINE` [Prisco et al. 2025]: C++ based, expected to be faster but still unstable

Evaluation strategy: medium term

- Fully numerical solution, using Pau's code
  - Expected to be faster (order of milliseconds per point)
  - Can be incorporated in a MC generator
- Requires some optimisations
  - Special functions to reduce redundancy (requires knowledge of the amplitude)
  - Polishing of the one-forms

Status of the calculation

- Planar families:
  - PL_1 ready
  - PL_2 and PL_3 missing only a few integrals
- Non-planar:
  - Planar sub-sectors from planar families
  - Expect only few genuinely new integrals

Conclusion and outlook

- The computation of the 2-loop amplitude requires a fast and reliable way to evaluate the Feynman integrals
- We are constructing DEs for the master integrals, addressing the complications related to the presence of elliptic Feynman integrals
- Next steps
  - Planar families
  - N_f - part of the amplitude
  - Non-planar families
  - 2-loop amplitude

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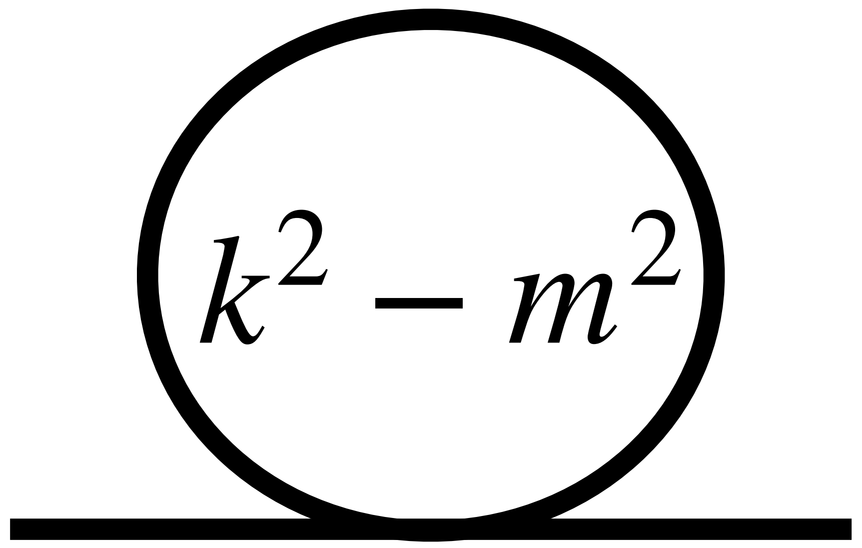
Thank you!

Backup slides

The solution space is redundant

$$I_j(\vec{x};\varepsilon) = \sum_{w=0}^{4} \varepsilon^w \, I_j^{(w)}(\vec{x})$$

- Not all $I_j^{(w)}(\vec{x})$ are independent!
- Example:



$$= 1 - \varepsilon \log(m^2) + \varepsilon^2 \left(\frac{\pi^2}{12} + \frac{1}{2} \log(m^2)^2 \right) + \mathcal{O}(\varepsilon^3)$$

- Goal: write the solution in terms of a set of algebraically independent functions
  - Faster evaluation of DEs
  - Analytic cancellation of poles
  - Significant simplification of the amplitude

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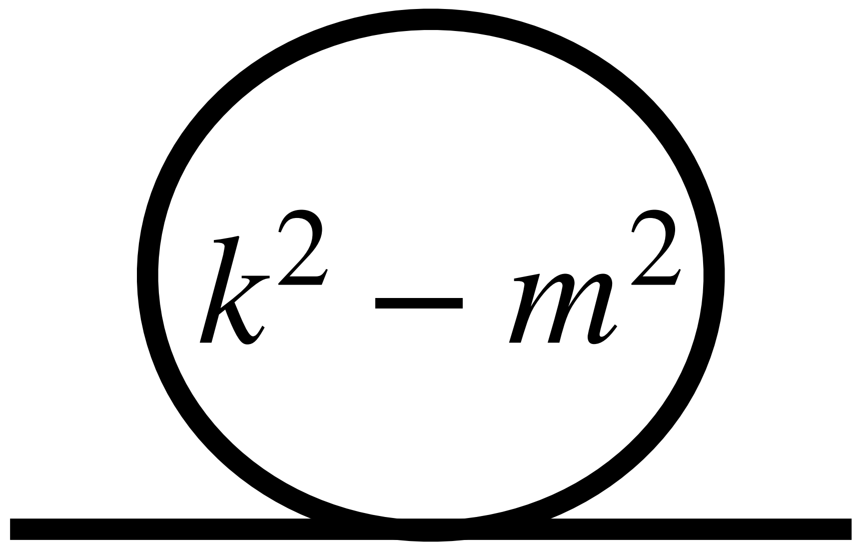

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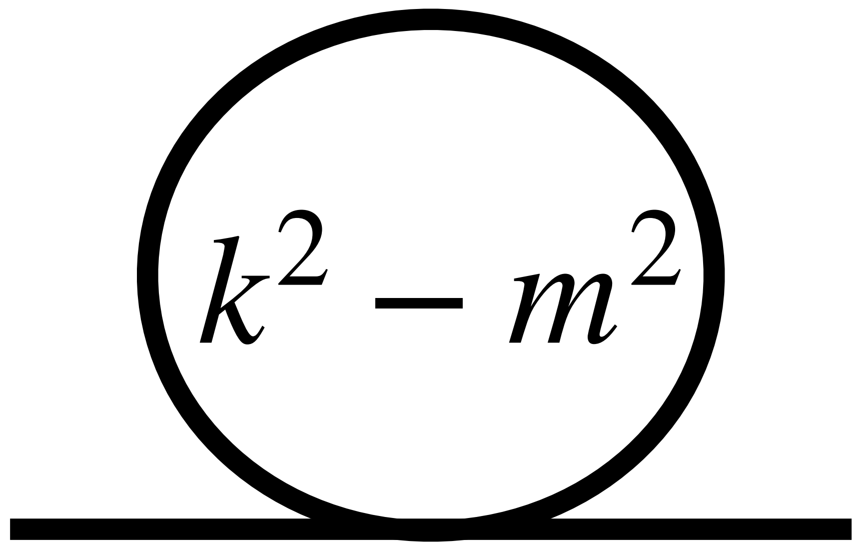

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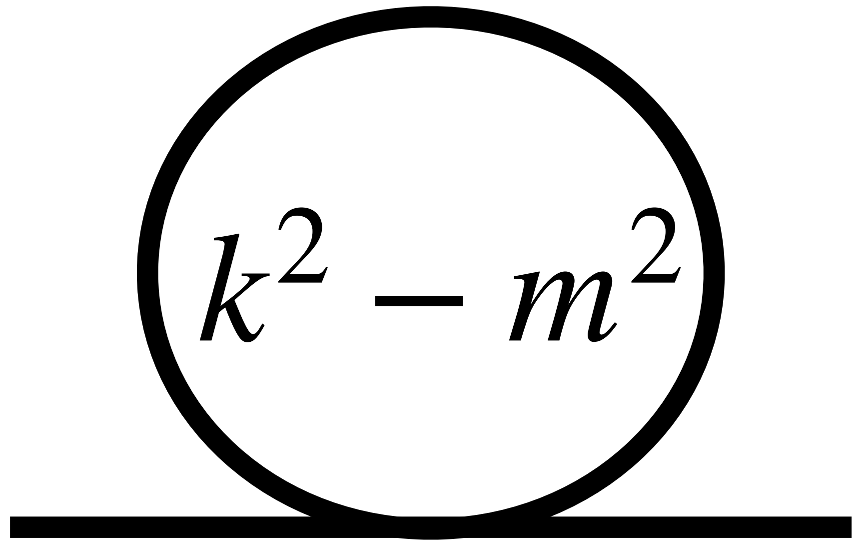
Transcendental constants



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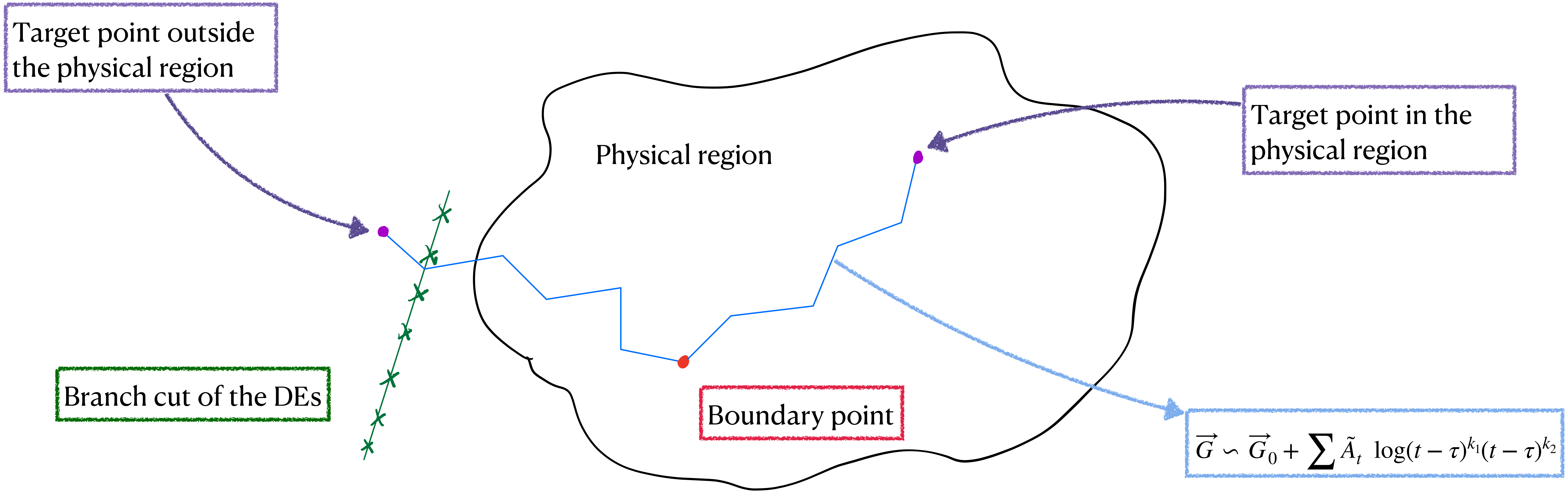
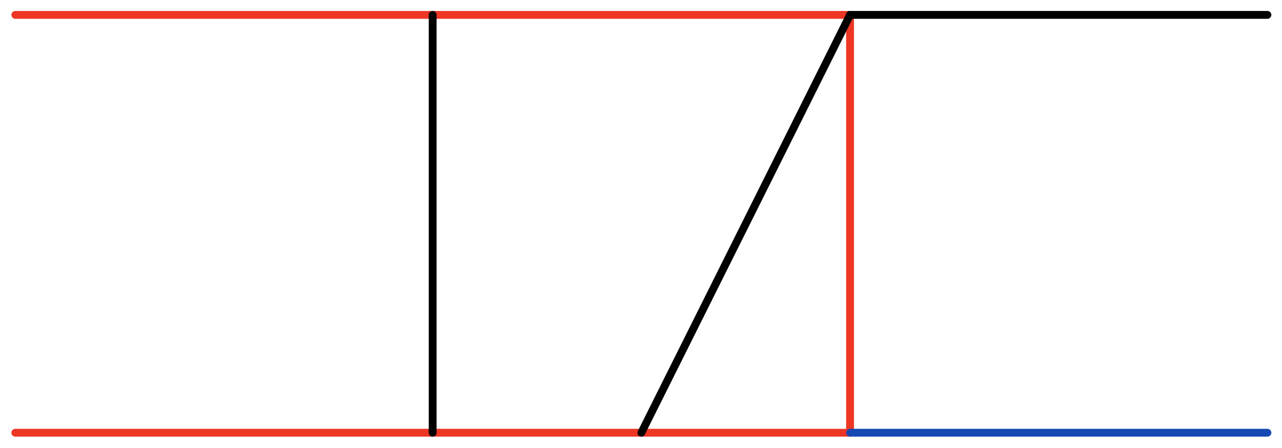
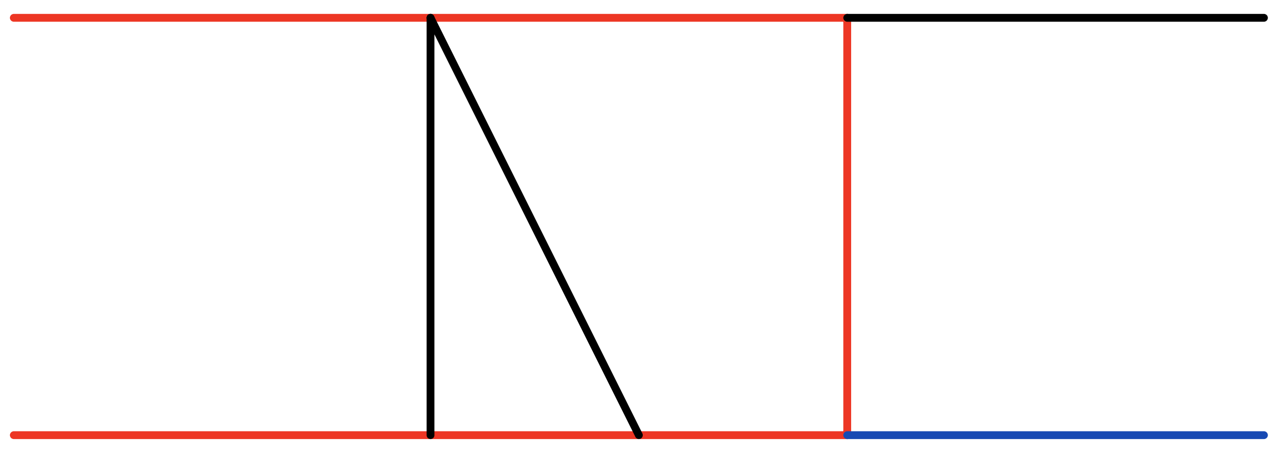
Generalised series expansion methods



The geometry is not obvious



Logarithmic



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